# On Calibration of the LIBOR Market Model to Caps Prices

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# 1. Introduction

The LIBOR Market Model, *in the following cases shortly LMM*, is a tool to price and hedge interest rate derivatives, which are functions of market forward-rates. It was developed by Miltersen, Sandmann & Sondermann (1997) [11], Brace, Gatarek & Musiela (1997) [4], Musiela & Rutkowski (1997) [12] and Jamshidian (1997) [9]. Main version of the LMM below is based on evolving LIBOR forward-rates. Unless specified otherwise in the following cases, the LMM refers to the LIBOR forward-rate model (LFM). Within LMM, *also called LIBOR forward-rate model (LFM) as just mentioned above*, all pricing is done using LIBOR forward-rates only. It can be used to price any instrument whose pay-off can be decomposed linearly into a set of LIBOR forward-rates, and it assumes that under a suitable choice of numeraire(s), the evolution of each LIBOR forward-rate is lognormal.

By contrast with models that evolve the instantaneous short rate (*Vasicek, Hull-White, Black- Karasinski models*) or instantaneous forward-rates (*Heath-Jarrow-Morton model*), which are not directly observable in the market, the objects, i.e. LIBOR forward-rates, modelled using the LMM are market-observable quantities. Modelling the stochastic behavior of the unobservable financial quantities results in difficulties with model calibration, because, the calibration of model to a set of market quantities requires a highly non-linear transformation of the dynamics of these unobservable quantities into dynamics of observable quantities. To do this, complicated numerical procedures are needed, calibration requires optimization techniques and the results are not always satisfactory. So, these models are fairly cumbersome to be calibrated to market observables. However, the LMM is easyly calibrated to market quoted forward-rate volatilities. The LMM internal model values can be matched to the actual market values exactly.

The LMM is consistent with the market standard approach for pricing caps using Black's formula. Previous to LMM, the main drawback of the interest rate models is that there was no interest rate dynamics compatible with Black's formula for caps. The Black model (*for anything- bonds, caplets, swaptions*) assumes that at the expiry of the option the underlying variable is lognormally distributed with constant variance. In particular, the Black model for caplets assumes that each LIBOR forward-rate is lognormally distributed with constant variance. This variance is in the form of Black implied volatility. As a result of this, *the importance of Black's model* is that the market quotes the prices of caps in terms of Black implied volatilities.

All mentioned above is the answer of the question: "*why the LMM is the so popular in the interest rate models?*".

In this project, firstly, the essential definitions, necessary to comprehend the topic, will be provided. Then, the LIBOR Market Model theory will be presented to account for the LIBOR forward-rate dynamics by means of some mathematical tools given before, and the Black formula for caplets will be derived, so as to give the equivalence between the LMM and Black's caplet prices. Lastly, the principal part of the project, the calibration of the LMM to caps prices will be focused on. In order to price the interest rate derivatives in the LMM, the LIBOR forward-rates, the instantaneous volatility of these rates and the instantaneous correlation between them are needed to be specified. Here, the latter two parameters are not directly observable in the market and must be estimated from existing market data in a process known as calibration. A good source of observable market data is the price of caplets. Caplet prices are quoted in terms of the implied Black volatilities. Owing to the fact that there is a relation between the implied Black volatility and the instantaneous volatility of the LIBOR forward-rates, the instantaneous volatility structure is vital for the calibration process in the LMM. In particular, the choice of the $\sigma_k(t)$ is motivated in order to assure the best possible price of caps while at the same time keeping the calibration simple. The intention is to explain the structure of the instantaneous volatility function for optimal calibration to caps. Furthermore, the instantaneous volatility structure as far as calibration is concerned will be discussed.

# 2. Definitions

## Definition 2.1 (*Zero-Coupon Bonds*)

A zero-coupon bond is essentially a bond that only pays its face value when it matures. It does not make periodic interest payments, or have so-called "coupons", hence the term zero-coupon bond. The zero-coupon bond with maturity date $T$, *also called T-maturity zero-coupon bond*, is a contract which guarantees its holder one unit of currency to be paid at the maturity date $T$, with no intermediate payments [5]. $P(t,T)$ is the price at time $t$ of the zero-coupon bond maturing at time T. The stochastic zero-coupon bond price, $P(t,T)$, can be defined with respect to the stochastic instantaneous risk-free interest rate $r(t)$ [19], [15]:

$$P(t,T) = \exp^{-\int_t^T r(s)ds} \tag{2.1}$$

Assuming that $r(t) > 0$ for all $t$ one can derive the following results [19]:

- $P(t,T) = 1, \forall t$
- $P(t,T) < 1, \forall t < T$
- $P(t,T) > 0, \forall t, T$
- $P(t,T_i) > P(t,T_j), \forall t, T_i > T_j$

Notice that the instantaneous interest rate $r(t)$ cannot be observed from the market. Instead, zero-coupon bond prices can be stripped from market bond quotations with some simple methodology [19] .

## Definition 2.2 (*The LIBOR*)

In [10], the LIBOR, or *the London Inter Bank Offered Rate* is the spot rate, *which is the rate given today, i.e. at time t, for borrowing or lending money until time* $T_n$, offered by banks for lending to other banks. The LIBOR is normally quoted as the rate for n-month loan where n typically is $3, 6, 9, ...$months. There are equivalents to LIBOR in other countries, for instance the STIBOR (Stockholm Inter Bank Offered Rate) and the Euroland equivalent EURIBOR.

Notice that "*Although the model is named after the LIBOR, it is the LIBOR implied forward rate, not the LIBOR that is modelled by LMM*".

**Definition 2.3 (*LIBOR Forward-Rate*)**

A LIBOR forward-rate is the interest rate at which one can contract to borrow money from a specified future time and for a specified length of time. We set $I_i := ]T_{i-1}, T_i]$. Then, $F_i(t)$ refers to the LIBOR forward-rate, contracted at t, for the period $[T_{i-1}, T_i]$, that is, the LIBOR forward-rate $F_i(t)$ is fixed (*or reset*) at $T_{i-1}$, which is called *reset date*, and is applied (*or valid*) on the period $I_i$ of the length $\tau(T_{i-1}, T_i) := T_i - T_{i-1}$ which is known as the tenor. The LIBOR forward-rate $F_i(t)$ is a stochastic variable for all $t < T_{i-1}$ [19].

***This rate is quoted as follows*[2]:**
We let $P_i(t) = P(t, T_i)$ denote the zero coupon bond price

1) At time t we sell one $T_{i-1}$-bond. This will give us $P(t, T_{i-1})$ unit of money.

2) We use this income to buy exactly $\frac{P(t,T_{i-1})}{P(t,T_i)}$ $T_i$-bond. Therefore, our investment at time t is equal to zero.

3) At time $T_{i-1}$, the $T_{i-1}$-bond matures, so we are obliged to pay out one unit of money.

4) At time $T_i$ the $T_i$-bonds mature at one unit of money a piece, so we will receive the amount $\frac{P(t,T_{i-1})}{P(t,T_i)}$ unit of money

5) The net effect of all this is that, based on a contract at t, an investment of one unit of money at time $T_{i-1}$ has yielded $\frac{P(t,T_{i-1})}{P(t,T_i)}$ unit of money at time $T_i$

6) As a result of this, at time $t$, we have made a contract guaranteeing a riskless rate of interest over the future interval $[T_{i-1}, T_i]$. Such a interest rate is called a forward rate.

We can compute the LIBOR forward-rate easily from the construction above. The return at time $T_i$ of 1 unit of money borrowed out at time $T_{i-1}$ is equal to 1 plus actual interest rate that one gets from $T_{i-1}$ to $T_i$, the LIBOR forward-rate multiplied by the tenor, $\tau_i F_i(t)$. It is given as the following:

$$1 + \tau_i F_i(t) = \frac{P(t, T_{i-1})}{P(t, T_i)} \tag{2.2}$$

If we leave the $F_i(t)$ alone in (2.2), we will obtain the LIBOR forward-rate over period $[T_{i-1}, T_i]$:

$$F_i(t) = F(t; T_{i-1}, T_i) = -\frac{P(t, T_i) - P(t, T_{i-1})}{\tau(T_{i-1}, T_i)P(t, T_i)} \tag{2.3}$$

where $P(t, T_i)$ is the time t price of a risk-free zero coupon bond maturing at time $T_i$ with nominal value 1.

In [19], unlike zero-coupon bonds, LIBOR forward rates themselves are not tradable assets, that is, their payoff can not be bought or sold in the market. However, the amount $F_i(t)P(t, T_i)$ can be written as

$$F_i(t)P(t, T_i) = \frac{1}{\tau(T_{i-1}, T_i)}[P(t, T_{i-1}) - P(t, T_i)] \tag{2.4}$$

Since the right hand side of (2.4) is the price of a traded asset (difference between two zero-coupon bonds, each with nominal value $[\tau(T_{i-1}, T_i)]^{-1}$), the left hand side must be also a traded asset [3]. That is to say, the right hand side of (2.4) is a portfolio of traded assets. So, the amount $F_i(t)P(t, T_i)$ is tradable itself.

## Definition 2.4 (*Interest-Rate Caps*)

In [18], an interest rate cap is a contractual arrangement where the grantor (*or seller*) has an obligation to pay cash to the holder (*buyer*) if the LIBOR forward-rate exceeds a mutually pre-negotiated level at some future date or dates. When cash is paid to the holder, the holder's net position is equivalent to borrowing at a rate fixed at that agreed level. This assumes that the holder of a cap agreement also holds an underlying asset (*such as deposit*) or an underlying liability (*such as a loan*). Finally, the holder is not affected by the agreement if the interest rate is ultimately more favorable to him than the agreed level. This feature of a cap agreement makes it similar to an option. Therefore, it is said that an interest rate cap is a serious of European call options on the LIBOR forward-rate that insure against rising interest rates. The company is afraid that LIBOR forward-rates will increase in the future, and wishes to protect itself whereby locking the payment at a maximum '*cap rate R*'. That is to say, *the interest-rate cap* is a derivative in which the holder receives payments at the end of each period in which the interest rate exceeds the agreed strike price (*cap rate, R*).

A caplet is a call option on a LIBOR forward-rate. The holder of a caplet at expiry has the right to enter into a forward contract to borrow money at an agreed interest rate R, known as strike price, from the expiry time until some future time.

In the market, caplets are not traded, they are always traded in the form of caps. A cap consists of multiple (different) caplets. Since a cap is split additively in caplets and each caplet can be priced separately, the cap prices can be calculated as the sum of the caplet prices.

In [2], [7] and [13], consider a debt for the $i$th LIBOR forward-rate, with a notional amount (*or nominal value*) N. A caplet on this debt will be called as "*a caplet on the ith LIBOR forward-rate*". The $i$th caplet expires at time $T_{i-1}$ and at expiry allows the holder to enter into a forward rate until time $T_i$. Suppose the strike rate (*or cap rate*) is R. The payoff received from the $i$th caplet at time $T_i$ is

$$C_i(T_i) = N\tau(T_{i-1}, T_i)(L(T_{i-1}, T_i) - R)^+ \tag{2.5}$$

where $L(T_{i-1}, T_i) := F(T_{i-1}; T_{i-1}, T_i) = F_i(T_{i-1})$ denotes the LIBOR spot rate (*or simply-compounded spot interest rate*) for the period $[T_{i-1}, T_i]$ at time $T_{i-1}$, and determined already at time $T_{i-1}$. Therefore, the amount $C_i(T_i)$ is determined at $T_{i-1}$ but not payed out until the time $T_i$ to the holder of the cap. Formally speaking, the caplet $C_i(T_i)$ is a call option on the underlying LIBOR spot rate. Here, the function $(.)^+ : \mathbf{R} \to \mathbf{R}$ is defined by $(x)^+ := max(x, 0)$ for $x \in \mathbf{R}$.

In [5], let the LIBOR forward rate indebted company has to pay at $T_{\alpha+1} < ... < T_\beta$. So, $T_\alpha < ... < T_{\beta-1}$ denotes the resettlement dates. As it is mentioned before, a Cap is a portfolio of the individual caplets, $C_{\alpha+1}(T_{\alpha+1}), ..., C_\beta(T_\beta)$. The discounted payoff at time $t$ of a caplet is defined as

$$D(t, T_i)N\tau(T_{i-1}, T_i)(L(T_{i-1}, T_i) - R)^+ \tag{2.6}$$

where N is notional amount (*or nominal value*), $D(t, T_i)$ is the (*stochastic*) discount factor between two time instants $t$ and $T_i$. The discount factor is the amount at time t that is "*equivalent*" to one unit of currency payable at time $T_i$ and is given by

$$D(t, T_i) := \frac{B(t)}{B(T_i)} = e^{(-\int_t^{T_i} r_s ds)} \tag{2.7}$$

where $B(t)$ is the value of a bank account at time $t \geq 0$ and $r(s)$ is the instantaneous interest rate (or *instantaneous spot rate, short rate*) at which the bank account accrues.

Notice that the evolution of the instantaneous interest rate $r(s)$ follows a stochastic process in time. As a consequence, the discount factor will be a stochastic process, that is, $D(t, T_i)$ is a random quantity at time $t$ depending on the future evolution of instantaneous rates $r(s)$ between $t$ and $T_i$.

The discounted payoff at time $t$ of a cap with first reset date $T_\alpha$ and payment dates $T_{\alpha+1} < ... < T_\beta$ is given by

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau(T_{i-1}, T_i)(L(T_{i-1}, T_i) - R)^+ \tag{2.8}$$

# 3. LIBOR Market Model Theory

In [14], consider a financial market "$M$" in which $N+1$ non-dividend-paying assets, namely $N+1$ bonds, are traded continuously from time 0 up to time $T_N$. A set of $N+1$ bonds maturities $\{T_i\}_{i=0}^{N}$ is given, with

$$0 < T_0 < T_1 < T_2 < ... < T_N < T$$

The maturity of the $i$th bond is $T_i$, $i = 0, 1, ..., N$. The price process of the $i$th bond is denoted by $P_i(t) = P(t, T_i)$.

Here, there is uncertainty as to what the future prices of these assets will be. This uncertainty will be modelled through a d-dimensional Brownian motion $W$ defined on its canonical probability space $(\mathbf{\Omega}, \mathbf{F}, \mathbf{P})$. The Filtration $\mathbf{F} = F(t) : 0 \leq t \leq T$ is the $\sigma$-field generated by $\sigma(W(s) : 0 \leq s \leq t)$ and the null sets of $\mathbf{F}$, that is, the prices, $P_i(t), i = 0, 1, ..., N$, of these assets can be modelled as Itô processes which are described by stochastic differential equations in which the uncertainty is given by a Brownian Motion $W$.

$$\begin{aligned}
\frac{dP_i(t)}{P_i(t)} &= \omega_i(t)dt + \beta_i(t)dW(t) \\
&= \omega_i(t)dt + \sum_{j=1}^{d} \beta_{ij}(t)dW_j(t),\ 0 \leq t \leq T_i, \\
P_i(0) &= b_{0,i}^{Market},\ i = 0, 1, ..., N \\
dW_i dW_j &= \rho_{ij} dt,\ i, j = 0, 1, ..., N
\end{aligned}$$

where $b_{0,i}^{Market}$ is the zero-coupon bond price observed in the market at time 0.

## 3.1 Equivalent Martingale Measure and Martingale Pricing

**Definition 3.1.1 (*Numeraire*)**

A numeraire is a price process $(N(t))_{t=0}^{T}$ of a non-dividend-paying asset (a sequence of random variables), which is strictly positive for all $t \in [0, T]$ [1], [5].

Numeraires can be used to express all prices in a financial market, that is, it is a reference asset which is chosen so as to normalize all other asset prices with respect to it.

Suppose that the asset price process $P_1(t) = P(t, T_1)$ is chosen as numeraire. The prices of other assets expressed in $P_1(t)$ are called as "*the relative prices (or normalized process)*" [2], [5], [15] and is denoted by

$$\widehat{P}_i(t) = \frac{P_i(t)}{P_1(t)},\ i = 0, 1, ..., N \tag{3.1}$$

**Definition 3.1.2 (*Equivalent Martingale Measure (EMM)*)**

In [7] and [10], let $(\mathbf{\Omega}, \mathbf{F}, \mathbf{P})$ denote the probability space as mentioned before. The set of Equivalent Martingale Measures is the set of probability measures, $\mathbf{Q}^*$, with the following properties:

- $\mathbf{Q}^*$ is equivalent to $\mathbf{P}$, i.e., $\mathbf{Q}^*(A) = 0$ if and only if $\mathbf{P}(A) = 0$, for every $A \in \mathbf{F}$
- The relative price processes $\widehat{P}_i(t), i = 0, 1, ..., N$ are martingales under $\mathbf{Q}^*$ for all i, i.e., for $s \leq t$, $E^*[\widehat{P}_i(t)|\mathbf{F_s}] = \widehat{P}_i(s)$.

An equivalent martingale measure will often be referred to as just "*a martingale measure*" or as "*an EMM*".

**Definition 3.1.3 (*Martingale Pricing*)**

In [2], [7] and [10], suppose that the equivalent martingale measure $\mathbf{Q^N}$ (*not necessarily unique*), for the priori given market $P_0(t), P_1(t), ..., P_N(t)$, connected with the numeraire $N$ is chosen such that the price of any attainable claim X normalized by N is a martingale under $\mathbf{Q^N}$, i.e.,

$$\frac{X(t)}{N(t)} = E^{\mathbf{Q^N}}[\frac{X(T)}{N(T)}|F_t],\ 0 \leq t \leq T.$$

As a result of the above property, the arbitrage free price process $\Pi(t;X)$ of any attainable claim X is given by martingale pricing formula:

$$\Pi(t;X) = N(t)E^{\mathbf{Q^N}}[\frac{X(T)}{N(T)}|F_t] \tag{3.2}$$

**Definition 3.1.4 (*Change of Numeraire*)**

Consider two different numeraires, N and M connected with the equivalent martingale measures $\mathbf{Q^N}$ and $\mathbf{Q^M}$, respectively. Since the prices must be independent of the choice of the numeraire [3], [10], one can state:

$$N(t)E^{\mathbf{Q^N}}[\frac{X(T)}{N(T)}|F_t] = M(t)E^{\mathbf{Q^M}}[\frac{X(T)}{M(T)}|F_t]$$

Let $G(T) = \frac{X(T)}{N(T)}$ be then we obtain that

$$E^{\mathbf{Q^N}}[G(T)|F_t] = E^{\mathbf{Q^M}}[G(T)\frac{N(T)/N(t)}{M(T)/M(t)}|F_t]$$

According to the above equality, the expectation of the martingale G under $\mathbf{Q^N}$ is equal to the expectation of G times the random variable $\frac{N(T)/N(t)}{M(T)/M(t)}$ under the measure $\mathbf{Q^M}$. This random variable is known as the *Radon-Nikodym Derivative*. So, the *Radon-Nikodym Derivative* that changes the equivalent measure $\mathbf{Q^M}$ into $\mathbf{Q^N}$ is given by

$$\frac{d\mathbf{Q^N}}{d\mathbf{Q^M}} = \frac{N(T)/N(t)}{M(T)/M(t)} \tag{3.3}$$

**Definition 3.1.5 (*Multi-dimensional Girsanov's Theorem*)**

In [1], let $W^M(t) = (W_1^M(t), W_2^M(t), \ldots, W_d^M(t))$, $t \geq 0$, be a d-dimensional vector of correlated Brownian motions under the same measure $\mathbf{Q^M}$, with correlation values given by matrix $\rho$.

If $k(t) = (k_1(t), k_2(t), \ldots, k_d(t))$, $t \geq 0$, is a vector of pre-visible stochastic processes adapted to the same filtration $\{F(t)\}_{0 \leq t \leq T}$, such that

$$\int_0^T k_i^2(s)ds < \infty \; a.s., \; i = 1, 2, \ldots, d \tag{3.4}$$

and $(\zeta(t) : 0 \leq t \leq T)$ is defined as

$$\zeta(t) = \exp\{\int_0^t k^\top(s)dW^M(s) - \frac{1}{2}\int_0^t \|k(s)\|^2 ds\} \tag{3.5}$$

then $\zeta(t)$ is the *Radon-Nikodym Derivative* of $\mathbf{Q^N}$ with respect to $\mathbf{Q^M}$, referred to as

$$\frac{d\mathbf{Q^N}}{d\mathbf{Q^M}} = \zeta(t)$$

and under the measure $\mathbf{Q^N}$ the process $W^N(t) = (W_1^N(t), W_2^N(t), \ldots, W_d^N(t)), t \geq 0$, is also a d-dimensional Brownian motion such that

$$W_i^N(t) = W_i^M(t) - \int_0^t k_i(s)ds, \; (0 \leq t \leq T), \; i = 1, 2, \ldots, d \tag{3.6}$$

A sufficient condition (also called as *Novikov's condition*) for $(\zeta(t))_{0<t<T}$ to be a martingale is defined by

$$E^M[\exp(\frac{1}{2}\int_0^T \|k(s)\|^2 ds)] < \infty \tag{3.7}$$

The main consequence of the Girsanov theorem is that "*when one changes the measures, the drift component is affected but the diffusion component remains unaffected.*"

**Definition 3.1.6 (*Forward Measure*)**

For a fixed T, *the T-forward measure* $\mathbf{Q^T}$ is defined as the equivalent martingale measure with the price at time t of the zero-coupon bond maturing at time T, $P(t, T)$, as numeraire . Under this martingale measure, the relative prices

$$\frac{P(t, T_i)}{P(t, T)}, \ i = 0, 1, ..., N$$

are martingale [1], [2], [10] and [20]. Furthermore, for any T-claim X, we have

$$\Pi(t; X) = P(t, T) E^{\mathbf{Q^T}}[X(T)|F_t] \tag{3.8}$$

where $E^{\mathbf{Q^T}}$ denotes integration with respect to $\mathbf{Q^T}$ [10].

## 3.2 LIBOR Market Model

### 3.2.1 *LIBOR Forward-Rate Dynamics in the LMM*

Let $t = 0$ be the current time. Consider a set of times $\Lambda = \{T_0, T_1, T_2, ..., T_N\}$ such that $0 \le T_0 < T_1 < T_2 < \ldots < T_N < T$. Pairs of expiry-maturity dates, $(T_{i-1}, T_i)$ for $i > 0$, are used to define a set of spanning LIBOR forward-rates. LIBOR forward-rates are given as $F_k(t) = F(t; T_{k-1}, T_k), k = 1, 2, ..., N$. The corresponding year fractions are $\tau_0, \tau_1, \tau_2, ..., \tau_M$. $\tau_i$ denotes the year fraction associated with the expiry-maturity pair $(T_{i-1}, T_i)$ representing the tenor of the LIBOR forward-rate and $\tau_i = T_i - T_{i-1}$. We set $T_{-1} := 0$. $\tau_0$ is the year fraction from settlement to $T_0$.

In a general case, under an arbitrary measure $Q$, the LIBOR forward-rate dynamics can be described by the following multi-dimensional stochastic differential equation [13], [16] :

$$\frac{dF(t)}{F(t)} = \mu(F, t)dt + S(t)dW^Q(t) \tag{3.9}$$

where

- ⋆ $\frac{dF(t)}{F(t)}$ denotes an N-dimensional column vector of percentage increments of LIBOR forward-rates.
- ⋆ $\mu(F, t)$ is N-dimensional column vector of drifts which may be functions of the LIBOR forward-rates themselves and time.
- ⋆ $dW^Q(t)$ is an N-dimensional column vector of correlated standard Brownian motions under the chosen measure $Q$.

- ⋆ $S(t)$ is a real $N \times N$ diagonal matrix, where the $i$th element in the diagonal $\sigma_i$, is equal to the instantaneous (percentage) volatility of the $i$th LIBOR forward-rate.

The correlation among the LIBOR forward-rates is not explicitly described in the LMM. However, the vector Brownian motion has correlation matrix $\rho$. The instantaneous correlations among the increments of the Brownian motions is given by an $N \times N$ correlation matrix as the following:

$$\rho = (\rho_{i,j})_{(i,j=1,2,\ldots,N)} \tag{3.10}$$

or equivalently

$$dW^Q(t)[dW^Q(t)]^\top = \rho dt \tag{3.11}$$

with element $\rho_{i,j}$ between Brownian motions $W_i^Q(t)$ and $W_j^Q(t)$,

$$dW_i^Q(t)dW_j^Q(t) = d < W_i^Q, W_j^Q >_t = \rho_{i,j}dt \tag{3.12}$$

where the upper indices in the Brownian motion denote under which measure being worked and the lower indices indicate the vector component.

Each LIBOR forward-rate is modelled by its own Brownian motion and associated instantaneous volatility function in (3.9) The Brownian motions are specified under the measure $Q$ which uniquely determines the drift vector. Also, the correlation matrix $\rho$ has full rank. As a result of this, the above equations describe the dynamics of the term structure with as many factors as LIBOR forward-rates.

The second formulation for the LIBOR forward-rate dynamics in matrix notation is given as the following stochastic differential equation [21], [16]:

$$\frac{dF(t)}{F(t)} = \mu(F,t)dt + \sigma(t)dZ^Q(t) \tag{3.13}$$

where the structure of $\frac{dF(t)}{F(t)}$ and $\mu(F,t)$ are same as given in (3.9), but now $Z^Q(t)$ is M-dimensional column vector of orthogonal (independent) Brownian motions under the measure $Q$ and $\sigma(t)$ is an $N \times M$ real matrix where the $\sigma_{ij}(t)$, the $(i,j)$th element, is the instantaneous volatility of the $i$th LIBOR forward-rate for the $j$th orthogonal Brownian motion at time $t$.

The number of orthogonal (independent) sources of uncertainty $M$, may differ from the number of LIBOR forward-rates $N$. Preferably, $M \ll N$. Each of the LIBOR forward-rates is affected by each of the $M$ Brownian shocks in

(3.13). The level of responsiveness to Brownian shocks $j$, $j = 1, 2, \dots, M$ for each LIBOR forward-rate $i$, $i = 1, 2, \dots, N$, is determined by factor $\sigma_{ij}(t)$. So, the instantaneous volatility of the LIBOR forward-rate $i$, $i = 1, 2, \dots, N$, is decomposed among $M$ Brownian motions. For each LIBOR forward-rate, this yields

$$\frac{dF_i(t)}{F_i(t)} = \mu_i dt + \sum_{j=1}^{M} \sigma_{ij}(t) dZ_j^Q(t), \ i = 1, 2, ..., N, \ M \leq N \tag{3.14}$$

where $F_i(t) = F(t; T_{i-1}, T_i)$ is the LIBOR forward-rate at time $t$ for the period $[T_{i-1}, T_i]$, $\mu_i$ is the drift parameter which can depend on both time and $i$th LIBOR forward-rate, $dZ_j^Q(t)$ is the $j$th orthogonal standard Brownian Motion at time $t$ under the measure $Q$ and $\sigma_{ij}(t)$ is as explained in (3.13).

Notice that in (3.13), the second formulation of the LIBOR forward-rates' dynamics, the correlation matrix will differ from the full-rank correlation matrix $\rho$ for $M \neq N$. If all LIBOR forward-rates are driven by a single source of uncertainty ($M = 1$), they have perfect instantaneous correlation.

The first formulation for the LIBOR forward-rate dynamics, (3.9), describes the yield curve evolution in terms of LIBOR forwad-rate specific Brownian shocks while the second formulation, (3.13), makes use of shocks affecting the entire yield curve and specifying the extent to which each LIBOR forward-rate is affected by each Brownian shock. The relationship between two formulations is the relationship between matrices $S(t)$ and $\sigma(t)$. The total instantaneous volatility of the LIBOR forward-rate $\sigma_i(t)$ and the $\sigma_{ij}(t)$ are related by

$$\sigma_i^2(t) = \sum_{j=1}^{M} \sigma_{ij}^2(t) \tag{3.15}$$

where $\sigma_i(t)$ is the $i$th element in the diagonal of the $N \times N$ diagonal matrix $S(t)$ and $\sigma_{ij}(t)$ is the $(i, j)$th element of the $N \times M$ matrix $\sigma(t)$.

Using the equality (3.15), if each $\sigma_{ij}(t)$ is divided and multiplied with the instantaneous volatility of the $i$th LIBOR forward-rate $\sigma_i(t)$ in (3.14), it will be obtained that

$$\begin{aligned}\frac{dF_i(t)}{F_i(t)} &= \mu_i dt + \sigma_i(t)\sum_{j=1}^{M}\frac{\sigma_{ij}(t)}{\sigma_i(t)}dZ_j^Q(t)\\ &= \mu_i dt + \sigma_i(t)\sum_{j=1}^{M}\frac{\sigma_{ij}(t)}{\sqrt{\sum_{j=1}^{M}\sigma_{ij}^2(t)}}dZ_j^Q(t)\\ &= \mu_i dt + \sigma_i(t)\sum_{j=1}^{M} b_{ij}(t)dZ_j^Q(t),\ i=1,2,...,N,\ M\le N\end{aligned}$$

where

$$b_{ij}(t) = \frac{\sigma_{ij}(t)}{\sqrt{\sum_{j=1}^{M}\sigma_{ij}^2(t)}}$$

and $b_{ij}(t)$ is the proportion of total volatility of $i$th LIBOR forward-rate attributable to $j$th Brownian shock.

Some comments about this equation are appropriate [13], [16] and [21]:

◃ The Brownian motion component $dW^Q(t)$ in (3.9) is replaced by $\mathbf{B}dZ^Q$, where B is an $N\times M$ matrix and $dZ^Q$ are $M$-dimensional column vector of the orthogonal (independent) Brownian motions.

$$\begin{aligned}\frac{dF(t)}{F(t)} &= \mu(F,t)dt + S(t)dW^Q(t)\\ &= \mu(F,t)dt + S(t)BdZ^Q(t)\\ &= \mu(F,t)dt + \sigma(t)dZ^Q(t)\end{aligned}$$

The components $(\{b_{ij}\}_{j=1}^{M})$ of the matrix B contain information about the correlation structure. In fact, $\rho = \mathbf{BB}^\top$. The correlation matrix of the Brownian motions specified in (3.9), $\rho$, is a real symmetric, positive-semidefinite matrix with ones in its diagonals.

◃ The model is driven by $M \le N$ orthogonal (independent) Brownian motions, where $M$ denotes the number of factors that drive the model. Consequently, the correlation matrix $\rho$ has rank $M$.

◃ $\sigma_i$ is the instantaneous volatility corresponding to the entries of the symmetric matrix $\mathbf{S}(t)$. It is related to the covariance that drive the

$M$ Brownian motions through

$$\sigma_i^2(t) = \sum_{k=1}^{M} \sigma_{ik}^2(t).$$

$\sigma_i(t)$ gives the total level of volatility of $i$th LIBOR forward-rate at time $t$.

In [13], [16] and [21], the model is characterized by

- The initial conditions $F_i(0),\ i = 1, 2, ..., N,$
- The choice of the functional form of the instantaneous volatility function,$\sigma_i(t)$. $\sigma_i(t)$ is obtained from the implied Black volatility, which is used to price caplets on the LIBOR forward-rate with expiry $T_i$, via the relationship

$$\sigma_{Black}^2(T_i) = \frac{1}{T_i} \int_0^{T_i} \sigma_i^2(t)dt \tag{3.16}$$

- The choice of the number of factors $M$, and
- The structure of the correlation matrix $\rho$, subject to the constraint that

$$\sum_{j=1}^{M} b_{ij}^2 = 1,\ i = 1, 2, \ldots, N \tag{3.17}$$

  For the case $M < N$, where the number of factors is smaller than the number of LIBOR forward-rates, the total volatility of the each LIBOR forward-rate must be fully recovered. This ensures correct pricing of caplets. Hence, the equality (3.17) must hold. That is to say, each row vector comprising the matrix B must have norm 1. As a result of this, correlation matrix $\rho = \mathbf{B}\mathbf{B}^{\top}$ is optimized and the diagonal entries of the correlation matrix equal to 1.

In arbitrage free markets, a price of any tradable asset divided by any reference asset (*or numeraire*, which would be chosen to be the risk neutral bank account $B(t)$ in the risk neutral pricing) is a martingale under the measure with respect to this reference asset [5]. Whereas arbitrage derivatives pricing theory is based on hedging with tradable assets, e.g. bonds, LIBOR forward-rates are not traded in markets, that is, one cannot go out and buy an amount of LIBOR forward rates. Owing to the fact that $F_k(t) = F(t; T_{k-1}, T_k)$ is not tradable, the process $\frac{F_k(t)}{B(t)}$ cannot be a martingale [19]. Unlike $F_k(t)$, the amount $F_k(t)P(t, T_k)$ is tradable (as mentioned in Def.2.3). Now, let the numeraire be the zero coupon bond $P(t, T_k)$. Then, we obtain that the price $F_k(t)P(t, T_k)$ divided by numeraire, $P(t, T_k)$, is simply $F_k(t)$ as the following:

$$\frac{F_k(t)P(t, T_k)}{P(t, T_k)} = F_k(t)$$

Therefore, $F_k(t)$ is a martingale under *the forward (adjusted) measure* $Q^k$ associated to the zero coupon bond price numeraire with maturity $T_k$, denoted by $P(t, T_k)$ (as mentioned in Def.3.1.6)[5], [8]. Its dynamics need to be driftless under $Q^k$ associated to $P(t, T_k)$ when $F_k(t)$ is modelled according to a diffusion process.

For every $k = 1, 2, ..., N$, the LIBOR forward-rate process $F_k(t)$ is a martingale under the corresponding forward measure $Q^k$, on the interval $[0, T_{k-1}]$ [2] and its dynamics is given by the stochastic differential equation [5]:

$$dF_k(t) = \underline{\sigma}_k(t)F_k(t)dW^k(t), \; t \leq T_{k-1} \tag{3.18}$$

where $W^k(t)$ is an N-dimensional column vector Brownian motion associated with the forward measure $Q^k$ with instantaneous covariance $\rho = (\rho_{i,j})_{i,j=1,2,...,N}$, $dW^k[dW^k]^\top = \rho dt$ and $\underline{\sigma}_k(t)$ is the horizontal N-vector instantaneous volatility at time t for the LIBOR forward-rate $F_k(t)$ such that

$$\underline{\sigma}_k(t) = \begin{bmatrix} 0 & 0 & \ldots & \sigma_k(t) & 0 & 0 \end{bmatrix}$$

with the only non-zero entry $\sigma_k(t)$ occurring at the $k$th position in the vector $\underline{\sigma}_k(t)$.

Notice that $F_i(t)P(t, T_k)$, $k \neq i$ is not a tradable asset and hence under the forward measure $Q^k$ only $F_k(t)$ is a martingale. As a result of this, under a certain measure only one LIBOR forward-rate is a martingale and this unique LIBOR forward-rate has driftless diffusion equation, but all other LIBOR forward-rates has non-zero drift terms in the diffusion equation under the same measure.

In a non-trivial pricing case, we are interested in evolving all LIBOR forward-rates under same measure. In order to do so, we have to know the diffusion equation for all the LIBOR forward-rates under that measure.

In [10], [19], assume that we are looking for the dynamics of the LIBOR forward-rate $F_k(t)$ under the forward measure $Q^{k-1}$ associated with the zero-coupon bond price numeraire, $P(t, T_{k-1})$, that is, we will move from the forward measure $Q^k$ to $Q^{k-1}$. A change of measure only changes the drift term whether the diffusion term remains unaffected. So as to determine the drift term under the forward measure $Q^{k-1}$, we must find the *Radon-Nikodym Derivative* (as mentioned in Def.3.1.4 in (3.3)) and the relation between the Brownian motions under the respective measures by using the Girsanov's theorem (as mentioned in Def.3.1.5).

According to the Change of Numeraire, the *Radon-Nikodym Derivative* is given by

$$\zeta(t) = \frac{dQ^{k-1}}{dQ^k} = \frac{\frac{P(t,T_{k-1})}{P(0,T_{k-1})}}{\frac{P(t,T_k)}{P(0,T_k)}} = \frac{P(0,T_k)P(t,T_{k-1})}{P(0,T_{k-1})P(t,T_k)} = \frac{P(0,T_k)}{P(0,T_{k-1})}(1+\tau_k F_k(t)).$$

Take differential of both sides:

$$\begin{aligned}
d\zeta(t) &= \frac{P(0,T_k)}{P(0,T_{k-1})} d(1+\tau_k F_k(t)) \\
&= \frac{P(0,T_k)}{P(0,T_{k-1})} \tau_k dF_k(t) \\
&= \frac{P(0,T_k)}{P(0,T_{k-1})} \tau_k \sigma_k(t) F_k(t) dW_k^k(t) \\
&= \frac{P(0,T_k)}{P(0,T_{k-1})} \tau_k \frac{1}{\tau_k} (\frac{P(t,T_{k-1})}{P(t,T_k)} - 1) \sigma_k(t) dW_k^k(t) \\
&= \frac{P(0,T_k)}{P(0,T_{k-1})} (\frac{P(t,T_{k-1})}{P(t,T_k)} - 1) \sigma_k(t) dW_k^k(t) \\
&= \frac{\zeta(t)}{\zeta(t)} \frac{P(0,T_k)}{P(0,T_{k-1})} (\frac{P(t,T_{k-1})}{P(t,T_k)} - 1) \sigma_k(t) dW_k^k(t) \\
&= \zeta(t) (\frac{1}{\frac{P(0,T_k)P(t,T_{k-1})}{P(0,T_{k-1})P(t,T_k)}}) \frac{P(0,T_k)}{P(0,T_{k-1})} (\frac{P(t,T_{k-1})}{P(t,T_k)} - 1) \sigma_k(t) dW_k^k(t) \\
&= \zeta(t) (\frac{P(t,T_k)}{P(t,T_{k-1})}) (\frac{P(t,T_{k-1})}{P(t,T_k)} - 1) \sigma_k(t) dW_k^k(t) \\
&= \zeta(t) (\frac{P(t,T_k)}{P(t,T_{k-1})}) \tau_k F_k(t) \sigma_k(t) dW_k^k(t) \\
&= \zeta(t) (\frac{1}{1+\tau_k F_k(t)}) \tau_k F_k(t) \sigma_k(t) dW_k^k(t) \\
&= (\frac{\tau_k F_k(t) \sigma_k(t)}{1+\tau_k F_k(t)}) \zeta(t) dW_k^k(t)
\end{aligned}$$

where $\sigma_k(t)$ is the instantaneous volatility of the LIBOR forward-rate $F_k(t)$ and $W_k^k(t)$ is the $k$th component of the N-dimensional column vector Brownian motion $W^k(t)$.

By means of the Girsanov's theorem given in Def.3.1.5, *Radon-Nikodym Derivative* is also defined as

$$\zeta(t) = \exp\{\int_0^t k(s) dW^k(s) - \frac{1}{2} \int_0^t k^2(s) ds\}$$

When the Itô formula is applied to the above equation, the *Radon-Nikodym Derivative* has the stochastic differential equation as the following:

$$d\zeta(t) = \zeta(t) k(t) dW_k^k(t).$$

Now, we must equate the differential equations of the *Radon-Nikodym Derivative* which we found from the two different aspect. Thereupon, we can obtain the following adapted process

$$k(t) = \frac{\tau_k F_k(t)\sigma_k(t)}{1+\tau_k F_k(t)}.$$

According to the Girsanov's theorem (as mentioned in Def.3.1.5), the relation between Brownian Motions in the differential form is given by

$$\begin{aligned} dW^{k-1}(t) &= dW^k(t) - k(t)dt \\ dW^{k-1}(t) &= dW^k(t) - \frac{\tau_k F_k(t)\sigma_k(t)}{1+\tau_k F_k(t)}dt \end{aligned}$$

It can now be written for multiple successive transforms:

$$dW^j(t) = dW^k(t) - \sum_{p=j+1}^{k} \frac{\tau_k F_k(t)\sigma_k(t)}{1+\tau_k F_k(t)}dt$$

where $j < k$. Similarly, the relation between Brownian Motions as a differential form for any $j > k$ is defined as

$$dW^j(t) = dW^k(t) + \sum_{p=k+1}^{j} \frac{\tau_k F_k(t)\sigma_k(t)}{1+\tau_k F_k(t)}dt.$$

Hence, the dynamics of LIBOR forward-rate $F_k(t)$ under forward-adjusted measure $Q^j$ associated to the numeraire $P(t,T_j)$ for $t \leq \min(T_j, T_{k-1})$ in the three cases $j < k$, $j = k$ and $j > k$ are, respectively,

$$\begin{aligned} j < k,\ t \leq T_j : dF_k(t) &= \sigma_k(t)F_k(t)\sum_{p=j+1}^{k}\frac{\rho_{k,p}\tau_p\sigma_p(t)F_p(t)}{1+\tau_p F_p(t)}dt + \sigma_k(t)F_k(t)dW_k^j(t) \\ j = k,\ t \leq T_{j-1} : dF_k(t) &= \sigma_k(t)F_k(t)dW_k^j(t) \\ j > k,\ t \leq T_{k-1} : dF_k(t) &= -\sigma_k(t)F_k(t)\sum_{p=k+1}^{j}\frac{\rho_{k,p}\tau_p\sigma_p(t)F_p(t)}{1+\tau_p F_p(t)}dt + \sigma_k(t)F_k(t)dW_k^j(t) \end{aligned}$$

where $W_k^j(t)$ is Brownian motion under $Q^j$. All of the above stochastic differential equations have a unique strong solution in case the coefficients $\sigma(.)$ are bounded [5].

### 3.2.2 *Black's Formula for Caplets*

The standard market approach for pricing caps is the Black-76 formula. Black's formula is based on mimicking the Black and Scholes model (1973) which was offered as a closed-form solution for the price of a standard European call/put option on equities [5]. In [19], the basic idea behind the Black and Scholes model is to assume the underlying instrument to follow lognormal dynamics with constant volatility and its current value as its expected value under the risk free numeraire. In 1976, Fisher Black proposed an extension to Black and Scholes model for option pricing on equities in order to price interest rate options as "*Black-76 model*". It is also known as Black's Formula.

In this part, the use of Black's Formula for pricing a single caplet will be presented.

We choose the forward measure $\mathbf{Q^i}$ (*the EMM with the zero-coupon bond process* $P(t, T_i)$ *as the numeraire*) in order to price the caplet at time t with reset date $T_{i-1}$ and payment date $T_i$, $C_i(t)$. The caplet is a tradable asset. Therefore, according to martingale pricing formula (as mentioned in Def.3.1.3), the price of a claim is the expected value of its payoff divided by the numeraire's terminal value and multiplied by the numeraire's current value. The martingale pricing formula yields

$$C_i(t) = E_t^{Q^i}[\tau_i P(t, T_i) \frac{(L(T_{i-1}, T_i) - R)^+}{P(T_i, T_i)}]$$

where $\tau_i = T_i - T_{i-1}$ [13]. The term $P(t, T_i)$ is known at time t and can thus be taken out from the expectation. It is also known that $P(T_i, T_i) = 1$, which leads to

$$\begin{aligned} C_i(t) &= P(t, T_i)\tau_i E_t^{Q^i}[(L(T_{i-1}, T_i) - R)^+] \\ &= P(t, T_i)\tau_i E_t^{Q^i}[(F_i(T_{i-1}) - R)^+] \end{aligned}$$

So as to compute the expectation $E_t^{Q^i}[(F_i(T_{i-1}) - R)^+]$, we must know the distribution of $F_i(T_{i-1})$.
In [8] and [14], owing to the fact that $F_i(t)$ is a martingale under *the forward (adjusted) measure* $\mathbf{Q^i}$ associated to the zero coupon bond price numeraire with maturity $T_i$, $P(t, T_i)$, the LIBOR forward-rate $F_i(t)$ is assumed to follow a process given by the stochastic differential equation:

$$dF_i(t) = \sigma_i(t) F_i(t) dW_i^i(t), \ t \leq T_{i-1} \tag{3.19}$$

where $W_i^i$ denotes a Brownian motion under the measure $\mathbf{Q^i}$. The stochastic differential equation of the LIBOR forward-rate $F_i(t)$ under the forward measure $\mathbf{Q^i}$,(3.19) can be solved explicitly and the solution is a Geometric Brownian motion. It is given as

$$F_i(t) = F_i(0)e^{\int_0^t \sigma_i(s)dW^i(s) - \frac{1}{2}\int_0^t \|\sigma_i(s)\|^2 ds}, 0 \leq t \leq T_{i-1} \tag{3.20}$$

which can be verified using Itô formula. We know that $\sigma_i$ is assumed to be deterministic. Therefore, the LIBOR forward-rate $F_i(t)$ is assumed to be lognormal distributed, which is the one of the assumptions of Black's formula.

In (3.20), we set the time $t := T_{i-1}$ and let $\nu_i^2 = \int_0^{T_{i-1}} \sigma_i^2(s)ds$. Then we can write

$$F_i(T_{i-1}) = F_i(0)e^{\int_0^{T_{i-1}} \sigma_i(s)dW^i(s) - \frac{1}{2}\int_0^{T_{i-1}} \|\sigma_i(s)\|^2 ds} = F_i(0)e^Z \tag{3.21}$$

where Z is an $\mathbf{F(T_{i-1})}$-measurable random variable which is normally distributed under the forward measure $\mathbf{Q^i}$ such that

$$Z \sim N(-\frac{1}{2}\nu_i^2, \nu_i^2) \ \ with \ \ \nu_i^2 = \int_0^{T_{i-1}} \sigma_i^2(s)ds.$$

In accordance with the distribution of $Z$, in [20], the lognormal distribution for $F_i(T_{i-1})$ is given as

$$\ln[F_i(T_{i-1})] \sim N(\ln[F_i(0)] - \frac{1}{2}\nu_i^2, \nu_i^2) \ \ with \ \ \nu_i^2 = \int_0^{T_{i-1}} \sigma_i^2(s)ds.$$

Now, the expectation $E_t^{Q^i}[(F_i(T_{i-1}) - R)^+]$ can be easily computed according to Black's formula which consists of the Black and Scholes pricing formula for a call option on an underlying instrument. Here, the underlying instrument is $F_i$, struck at $R$, with maturity $T_{i-1}$, with 0 constant "risk-free rate" and instantaneous percentage volatility $\sigma_i(t)$ [6].

***Equivalence between LMM and Black's caplet prices*[5],[6]:**

The $T_{i-1} - caplet$ is a contract which is determined at time $T_{i-1}$ and pays at time $T_i$.

When $(\nu_{T_{i-1}-caplet}^{Black})^2 := \frac{1}{T_{i-1}}\int_0^{T_{i-1}} \sigma_i^2(t)dt$ satisfies, the LMM and the Black-76 model $T_{i-1} - caplet$ prices will be equal. That is to say, the price of the $T_{i-1}$-caplet implied by the LMM coincides with the corresponding Black caplet formula:

$$\begin{aligned} Cpl^{LMM}(0, T_{i-1}, T_i, R) &= Cpl^{Black}(0, T_{i-1}, T_i, R, \nu_i) \\ &= P(0, T_i)\tau_i Bl(R, F_i(0), \nu_i) \end{aligned}$$

$$\begin{aligned} Bl(R, F_i(0), \nu_i) &:= E_t^{Q^i}[(F_i(T_{i-1}) - R)^+] \\ &:= F(0, T_{i-1}, T_i)N[d_1(R, F_i(0), \nu_i)] - RN[d_2(R, F_i(0), \nu_i)] \\ d_{1,2}(R, F, \nu) &:= \frac{\ln \frac{F}{R} \pm \frac{\nu^2}{2}}{\nu} \end{aligned}$$

where

$$\begin{aligned} \nu_i^2 &= T_{i-1}(\nu_{T_{i-1}-caplet}^{Black})^2 \\ (\nu_{T_{i-1}-caplet}^{Black})^2 &:= \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i^2(t)dt. \end{aligned}$$

and $N : \mathbf{R} \to [0, 1]$ is the standard normal distribution function:

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy, \; for \, x \in \mathbf{R}.$$

$\sigma_i^2(t)$ is the instantaneous percentage volatility at time t for the LIBOR forward-rate $F_i(t)$. The quantity $\nu_{T_{i-1}-caplet}^{Black}$ is the implied Black volatility for the $T_{i-1}$-caplet associated with the forward rate $F_i(t)$ resetting at time $T_{i-1}$ and it is implicitly defined as the square root of the average percentage variance of the forward rate $F_i(t)$ for $t \in [0, T_{i-1})$[14], [17]. The Black formula returns the market quoted price of the caplet with the *implied Black volatility of a caplet*

***Cap is quoted in the market [5], [19]:***

The market quotes the prices of caps, not the prices of caplets. A common practise in the market is to quote cap prices as LIBOR forward-rate volatilities. These volatilities can be converted to actual prices with Black's formula, that is, cap prices are expressed in terms of implied Black volatilities. The caplet volatilities may be obtained from the cap volatilities quoted in the market by using a boot-strapping algorithm. Then, they can be used as market inputs.

The market quotes volatilities for caps with first reset date either in three months ($T_0$ equals to three months, $\alpha = 0$ and all other T's equally three-months spaced) or in six months ($T_0$ equals to six months, $\alpha = 0$ and all other T's equally six-months spaced). We set $\mathcal{T}_j = [T_0, T, ..., T_j]$ for all j.

There is an equation between the market price $\mathbf{Cap}^{MKT}(0, \mathcal{T}_j, R)$ of the cap with $\alpha = 0$ and $\beta = j$ (as mentioned in Def.2.4, it means that the resettlement dates are given as $T_0 < T_1 < ... < T_{j-1}$) and the sum of the first j caplets prices as

$$\mathbf{Cap}^{MKT}(0, \mathcal{T}_j, R) = \sum_{i=1}^{j} \tau_i P(0, T_i) Bl(R, F_i(0), \sqrt{T_{i-1}} \nu_{T_j - cap}) \tag{3.22}$$

where $\nu_{T_j-cap}$ is the average LIBOR forward-rate volatility of a cap with first reset date $T_0$ and last payment date $T_j$. So, the $\nu_{T_j-cap}$ are called sometimes "*forward volatilities*". Market quotes the cap prices as the corresponding volatility $\nu_{T_j-cap}$. The above equation is solved in $\nu_{T_j-cap}$ by the market and market quotes $\nu_{T_j-cap}$ as annualized and in percentages.

If two overlapping caps have different volatility quotations (as they always do), it implies two different instantaneous volatilities for the overlapping caplets, that is, when the same caplets concur to a different cap, their average volatility is changed. Despite this, same average-volatility value $\nu_{T_j-cap}$ is put for all overlapping caplets to the $T_j$-maturity cap up to j in the pricing methodology (3.22). As a result of this, there is an inconsistency in caplet volatilities. So as to link a unique volatility to each LIBOR forward-rate and still recover market quoted cap prices, the following equation must hold

$$\sum_{i=1}^{j} \tau_i P(0, T_i) Bl(R, F_i(0), \sqrt{T_{i-1}} \nu_{T_j - cap}) = \sum_{i=1}^{j} \tau_i P(0, T_i) Bl(R, F_i(0), \sqrt{T_{i-1}} \nu_{T_{i-1} - caplet})$$

where $\nu_{T_{i-1}-caplet}$ is the annualized volatility of a $T_{i-1} - caplet$. $\nu_{T_{i-1}-caplet}$ are called sometimes "*forward forward volatilities*". Different average volatilities $\nu_{T_{i-1}-caplet}$ are assumed for different caplets concurring to the $T_j$-maturity cap.

Since the cap volatilities are visible in the market for each $j = 1, 2, ..., N$, a simple stripping algorithm can be used to recover the $\nu_{caplet}$'s from the market quoted $\nu_{cap}$'s based on the last equality applied to $j = 1, 2, 3, ...$ . The market quoted cap volatilities can be converted into caplet volatilities by means of this methodology.

***Correlations have no impact on caps prices:***

The correlation is irrelevant when pricing products which depend only on a individual LIBOR forward-rates [13]. Since, there are no expectations involving two or more LIBOR forward-rates at the same time while pricing [5]. A cap is split additively in caplets and each caplet only depends on a single LIBOR forward-rate. Therefore, the instantaneous correlation $\rho_{i,j}$ between the LIBOR forward-rates $F_i(t)$ and $F_j(t)$ does not affect the overall cap price [19]. Only the $\sigma$'s have impact on cap prices. As a result of this, the calibration to cap prices is relatively straightforward.

# 4. Calibration to the Caps Prices

Here, the calibration is the computation of the parameters of the LMM, $\sigma_i(.), i = 1, ..., M$: *instantaneous volatilities of LIBOR forward rates*, so as to match as closely as possible model derived prices/values to market observed prices/values of actively traded securities, namely caps. Re-calibrating the model day after day to the future market prices is needed. The procedure of re-calibrating everyday the model to the current market is essential.

## 4.1 Instantaneous Volatility Structure

We will focus on the question "*How is the instantaneous volatility function $\sigma_k(t)$ appearing in the stochastic differential equation describing the forward rates determined?*". Determining the instantaneous volatility function is an important aspect in the specification of the model.

In [21], the success of the LMM implementation depends on the quality of the instantaneous volatility function specification. The form of the instantaneous volatility function should be financially intuitive with associated parameters, which have a clear econometric interpretation and can be related to market observables, and allow the term structure of the volatility to maintain its general shape through time. Moreover, it should be computational (or analytically) feasible and provide sufficient flexibility as to the exact shape of the humped curve, but not allow to many degrees of freedom (free parameters), which could result in over fitting to observed market data.

There are several parametric and non-parametric functions for the instantaneous volatility function $\sigma_k(t)$. We will give the more important ones in proportion to the rest of them.

In [5], it is often assumed that the LIBOR forward-rate $F_k(t)$ has a piecewise-constant instantaneous volatility as

$$\sigma_k(t) = \sigma_{k,\beta(t)},\ t > T_{-1} = 0,$$

where in general $\beta(t) = m$, *which is the index of the first LIBOR forward rate* $F_{\beta(t)}$ *that has not expired by* $t$ *and all preceding LIBOR forward rates* $F_{\beta(t)-1}, F_{\beta(t)-2}, \ldots$ *have expired by* $t$, if $T_{m-2} < t \leq T_{m-1},\ m \geq 1$, so that $t \in (T_{\beta(t)-2}, T_{\beta(t)-1}]$. In particular case $t = T_{-1} = 0$, we set $\sigma_k(0) = \sigma_{k,1}$.

**TABLE 1**

| Instant. Vols | Time:$t \in (0, T_0]$ | $(T_0, T_1]$ | $(T_1, T_2]$ | ... | $(T_{N-2}, T_{N-1}]$ |
|---|---|---|---|---|---|
| Fwd rate:$F_1(t)$ | $\sigma_{1,1}$ | 0 | 0 | ... | 0 |
| $F_2(t)$ | $\sigma_{2,1}$ | $\sigma_{2,2}$ | 0 | ... | 0 |
| $\vdots$ | ... | ... | ... | ... | ... |
| $F_N(t)$ | $\sigma_{N,1}$ | $\sigma_{N,2}$ | $\sigma_{N,3}$ | ... | $\sigma_{N,N}$ |

Under the general piecewise constant (GPC) assumption, the instantaneous volatilities in a matrix as in the form presented above, *TABLE 1*. Each LIBOR forward-rate has a unique instantaneous volatility at each time interval. Notice that all the instantaneous volatilities in the upper triangle part of the matrix are dead. They belong to forward rates that have already been fixed. That is to say, after time $T_i$, the $i$th LIBOR forward-rate has expired and its volatility is set to zero [3].

When GPC parameterization is taken into account, the number of volatility parameters are so many as it is seen in the *Table 1*. Thus, several alternative assumptions can be made so as to reduce the number of volatility parameters and generate a desirable future evolution on the volatility term structure. In [21], *the qualitative shape of the volatility term structure* displays a humped shape, with relatively low volatility on the short end, a peak around the 18 month maturity and then decreasing volatility towards the long end of the maturity spectrum, with a possible flattening out of volatilities. The general shape of the volatility term structure remains the same through time, that is, volatilities of caps with different terms to maturity maintain the same relative magnitudes through time. There may be short periods when the term structure displays anomalous behaviour, taking on a generally monotonically decreasing shape. Here, only two of the alternative assumptions will be given.

In [5] and [6], the first one is "*separable piecewise constant*" (SPC) assumption. In SPC parametrization, instantaneous volatilities consist of the product of a structure which only depends on the time-to-maturity ($T_k - T_{\beta(t)-1}$), (the $\Psi$'s part), by a structure which only depends on the maturity $T_k$, (the $\Phi$'s part). It is denoted as

*SPC parameterization*: $\sigma_k(t) = \sigma_{k,\beta(t)} := \Phi_k \Psi_{k-(\beta(t)-1)}$ for all $t$.

**TABLE 2**

| Instant. Vols | Time:$t \in (0, T_0]$ | $(T_0, T_1]$ | $(T_1, T_2]$ | ... | $(T_{N-2}, T_{N-1}]$ |
|---|---|---|---|---|---|
| Fwd rate:$F_1(t)$ | $\Phi_1\Psi_1$ | 0 | 0 | ... | 0 |
| $F_2(t)$ | $\Phi_2\Psi_2$ | $\Phi_2\Psi_1$ | 0 | $\ldots$ | 0 |
| $\vdots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $F_N(t)$ | $\Phi_N\Psi_N$ | $\Phi_N\Psi_{N-1}$ | $\Phi_N\Psi_{N-2}$ | $\ldots$ | $\Phi_N\Psi_1$ |

The SPC parameterization has the potential for maintaining the qualitative shape of the term structure of volatilities in time through its $\Psi$ part. If the $\Phi$'s are all equal, the term structure of volatility remains unchanged as time passes and in particular it maintains the hump over time. When the $\Phi$'s are not all equal, can change this, but if we make sure that the $\Phi$'s remain sufficiently close to each other, the qualitative behaviour will not be affected and the hump in the evolution of the term structure in time will be preserved.

Moreover, SPC parameterization generally does not result in a rank-one integrated covariance matrix, not even in the case of perfect instantaneous correlations $\rho$ all equal to one. However, it is rich enough to allow a satisfactory calibration to market data.

In [16], Rebonato (1999) proposed that basically, there are three functions which comprise the instantaneous volatilities of the LIBOR forward rates. These are given as the following forms:

1. Time to maturity depending part (or time-homogenous part): $f(T_{k-1} - t)$
2. Maturity depending part (or a complete function of the individual forward rate): $g(T_k)$
3. Calender time depending part (or the part depending only on current time $t$): $h(t)$

The most popular choice for the instantaneous volatilities consists of combinations of these proposed functional forms. Thus, the instantaneous volatilities are given by

$$\sigma_k(t) = f(T_{k-1} - t)g(T_k)h(t).$$

This general form displays several positive features, from both the financial and the computational perspective.

Based on the above functions, Rebonato suggests that "*linear-exponential form*" (LE), the second alternative assumption mentioned above [5],[6].

*LE formulation*:

$$\begin{aligned} \sigma_k(t) &= \Phi_k \Psi(T_{k-1} - t; a, b, c, d) \\ &:= \Phi_k([a(T_{k-1} - t) + d]e^{-b(T_{k-1}-t)} + c) \end{aligned}$$

Here, $f(T_{k-1} - t)$ and $g(T_k)$ refers to $\Psi$, which only depends on time to maturity, and $\Phi$, which only depends on maturity, respectively. $\Psi$ is locally altered for each maturity $T_k$ by $\Phi$. Also, the function $h(t)$ equals to 1.

LE formulation is the analogue of the SPC parameterization. The term structure remains the same as time passes, and in particular it can maintain its humped shape if initially humped and if all $\Phi$'s are not too different from one another.

In accordance to LE formulation, Rebonato accounts for these functions as the following.

***Time-homogenous part,*** $f(T_{k-1} - t)$ ***[16], [21] :***

Rebonato (1999) put forward three criteria for the function $f(T_{k-1} - t)$:

i. Owing to the fact that the term structure of volatilities should be as time homogenous as possible, the time homogeneous component dominates in determining the instantaneous volatilities. Thus, this function should be flexible enough to be able to produce both the empirically observed "humped" and monotonically decreasing shapes.

ii. Function parameters should have clear econometric interpretation, so as to allow a "sanity check" of calibration .

iii. Pricing of caplets (via the variance) and the evaluation of covariance terms requires integration of the square of this function. Thus, the function should afford easy analytical integration of its square in order to allow fast calculation of forward rate variance and covariance.

In direct proportion to above items, Rebonato introduces the function $f(T_{k-1}-t)$ as

$$f(T_{k-1}-t) = ([a(T_{k-1}-t)+d]e^{-b(T_{k-1}-t)}+c).$$

So as to preserve the short and long time behavior and the humped form of the term structure of volatilities one may not choose the parameters $a,\ b,\ c$ and $d$ completely free. For the purpose of fitting all the observed market caplet volatilities, the parameters $a,\ b,\ c,\ d$ are estimated by a least-square fit [3] and the following conditions must be satisfied for a well behaved instantaneous volatility structure.

* $d+c>0$
* $c>0$
* $b>0$

Furthermore, the parameters $a,\ b,\ c,\ d$ have the following financial interpretation:

- ◇ When $\delta = T_{k-1}-t \longrightarrow \infty$ (namely, $\delta$ tends to large values), $f(T_{k-1}-t) \longrightarrow c$. Hence, $c$ has to be connected with the very-long maturity volatilities. This implies that $c>0$, $b>0$.

- ◇ When $\delta = T_{k-1}-t \longrightarrow 0$, (namely, $\delta = T_{k-1}-t$ tends to zero), $f(T_{k-1}-t) \longrightarrow (d+c)$. In that case, the instantaneous and average volatilities tend to coincide. Therefore, the quantity $(d+c)$ should be approximately equal to shortest-maturity implied volatilities. This results in $(d+c)>0$.

- ◇ The location (on the time-to-maturity axis) of the hump is determined by $f'(\delta)=0$. That is to say, the first derivative of the time homogeneous part of LE formulation, $\Psi = f(T_{k-1}-t)$, with respect to $\delta = T_{k-1}-t$ gives the location of the hump. *The first derivative of the function $f(T_{k-1}-t)$ with respect to $\delta$*:

$$\begin{aligned} \frac{d}{d\delta}[f(\delta)] &= (a-bd-ba\delta)\exp^{-b\delta}=0 \\ \delta &= \frac{(a-bd)}{ba} \end{aligned}$$

  With regard to the result obtained from the derivation, we can make the following comments

- the extremum of the instantaneous volatility function (the top of the hump) in the LE formulation is reached at $\frac{(a-bd)}{ba}$, which is a local maximum for $a > 0$. If $a < 0$, no maximum occurs. Additionally, empirical evidence shows that the hump in the volatility curve is found around the one year point, hence $\frac{(a-bd)}{ba} \approx 1$.
- The curve in the "normal" state should have a positive initial slope, hence for $b > 0$, $f'(0) > 0$ implies $d < \frac{a}{b}$.

***Calender time depending part,*** $h(t)$ ***[16], [21]:***

There is no clear intuition to what qualitative features which should be looked for in the purely time dependent function $h(t)$. The choice of this function is more subjective than others. If the function is allowed to have too much freedom, it can pick up undesirable noise in the market term structure of volatilities. If it is too simple or too rigid, it might miss some of the important features which it is supposed to capture. Thus, Rebonato recommends a function of the type

$$h(t) = [\sum_{i_1}^{n} \epsilon_i \sin(\frac{t\pi i}{M} + \epsilon_{i+1})] \exp(-\epsilon_{n+1} t)$$

where $n$ is the number of free parameters (recommended to be as small as 2 or 3) and $M$ is the maturity of the longest caplet. This function is a linear combination of a small number of sine waves (where phases and amplitudes are optimized to the available data) multiplied by a exponential decaying term (with optimized decay constant to obtain an acceptable fit).

The easiest option is to set the function $h(t)$ to one so as to avoid picking up too much noise in the calibration. Alternatively, one can again use piecewise constant time dependency.

***Maturity depending part,*** $g(T_k)$***[16], [21] and [10]:***

The forward rate depending function $g(T_k)$ is the final smoothing function ensuring perfect pricing of today's market caplets (perfect fitting to today's volatility term structure). The function $g(T_k)$ does not need to be continuous on the ground that there are finite number of forward rates (also maturities $T_k$) in LMM framework. The value of the function $g(T_k)$ is set for each LIBOR forward-rate $k = 1, 2, \ldots, N$ in LMM by requiring that the model caplet prices fit the market data exactly. The function $g(T_k)$ allow a possibility for a perfect calibration and is therefore very useful. Its general form is given by

$$g(T_k) = 1 + \upsilon_k,\ 0 \le k \le N - 1$$

where N is the total number of the LIBOR forward-rate maturities, the $\upsilon_i$'s should all be approximately zero. *if the calibration is done correctly*, the function $g(T_k)$ is close to one

As a conclusion, in the LE formulation, in order to preserve time-homogeneity, it is important to assure that the $\Phi_k$'s are as close as possible to one. By means of putting $\Phi_k = 1$, the LE formulation is clearly time-homogenous and displays, for suitable choices of parameter set *(a,b,c,d)*, a nicely humped term structure of volatility and LE formulation gives a good fit to observed data from the market.

## 4.2 The Implication of The Instantaneous Volatility Structures As Far As Calibration is Concerned

The calibration procedure intends to assure that the models instantaneous volatility function resembles the implied Black volatility as good as possible [10].

Now, we will look into the instantaneous volatility structures, which are mentioned in 4.1, as far as calibration is concerned.

* In general, implied Black $T_{k-1} - caplet$ volatilities based on GPC parameterization are denoted [5] by

$$
\begin{aligned}
(\nu^{Black}_{T_{k-1}-caplet})^2 &= \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma^2_{k,\beta(t)} dt \\
&= \frac{1}{T_{k-1}} \sum_{j=1}^{k} (T_{j-1} - T_{j-2}) \sigma^2_{k,j}.
\end{aligned}
$$

* In [5] and [6], let the piecewise constant volatilities $\sigma_{k,\beta(t)}$ follow the SPC parameterization. Then, we obtain the following equality

$$
\begin{aligned}
T_{k-1}(\nu^{Black}_{T_{k-1}-caplet})^2 &= \int_0^{T_{k-1}} \sigma^2_{k,\beta(t)} dt \\
&= \int_0^{T_{k-1}} (\Phi_k \Psi_{k-(\beta(t)-1)})^2 dt \\
&= \Phi_k^2 \sum_{j=1}^{k} (T_{j-1} - T_{j-2}) (\Psi_{k-j+1})^2.
\end{aligned}
$$

If the squares of the implied Black $T_{k-1}$-caplet volatilities $(\nu^{MKT}_{T_{k-1}-caplet})^2$ are inferred from the market data, the parameters $\Phi$ can be defined in terms of the parameters $\Psi$ as

$$\Phi_k^2 = \frac{T_{k-1}(\nu^{MKT}_{T_{k-1}-caplet})^2}{\sum_{j=1}^{k}(T_{j-1}-T_{j-2})(\Psi_{k-j+1})^2}.$$

As a result of this, the caplet prices are incorporated in the model by determining the $\Phi$'s in terms of the $\Psi$.

* In [5] and [6], when the LE formulation is assumed for instantaneous volatilities, we obtain that

$$\begin{aligned} T_{k-1}(\nu^{Black}_{T_{k-1}-caplet})^2 &=: \int_0^{T_{k-1}} (\Phi_k \Psi(T_{k-1}-t;a,b,c,d))^2 dt \\ &= \Phi_k^2 \int_0^{T_{k-1}} (\Psi(T_{k-1}-t;a,b,c,d))^2 dt \\ &= \Phi_k^2 \int_0^{T_{k-1}} ([a(T_{k-1}-t)+d]e^{-b(T_{k-1}-t)}+c)^2 dt \\ &= \Phi_k^2 I^2(T_{k-1};a,b,c,d). \end{aligned}$$

The $\Phi$'s parameters can be used to calibrate caplet volatilities. If the squares of the implied Black $T_{k-1}$-caplet volatilities $(\nu^{MKT}_{T_{k-1}-caplet})^2$ are read from the market, the caplet volatilities are incorporated in the model by means of expressing the parameters $\Phi$ as functions of the parameters a, b, c, d. It is given by

$$\Phi_k^2 = \frac{T_{k-1}(\nu^{MKT}_{T_{k-1}-caplet})^2}{I^2(T_{k-1};a,b,c,d)}.$$

we calibrate the $f(T_{k-1}-t) = \Psi(T_{k-1}-t;a,b,c,d)$ part of the instantaneous volatility function in LE formulation. This is done by putting $g(T_k) = \Phi_k = 1,\ for\, all\, k,\ k = 1,2,...,N$, and performing the following least-square minimization of the variables a, b, c, d [10] :

$$\min_{s.t.} \sum_{\substack{k=1 \\ cd+c>0 \\ a>0 \\ b>0 \\ c>0 \\ d+c\approx smv \\ c\approx lmv \\ 0<\frac{(a-bd)}{ba}<const.}}^{N} \|(\sigma^{Black}_{T_{k-1}-caplet})^2 T_{k-1} - \int_0^{T_{k-1}} f^2(T_{k-1}-s)ds\|$$

smv and lmv are constants consistent with the Black-76 implied short and long maturity volatilities and const. is the upper bound on the location of the hump.

In [10], after applying best possible least-square fit of the function $f(T_{k-1}-t)$, one may use the function $g(T_k) = \Phi_k$ to assure perfect caplet pricing by letting

$$\Phi_k^2 = \frac{(\sigma^{Black}_{T_{k-1}-caplet})^2 T_{k-1}}{\int_0^{T_{k-1}} f^2(T_{k-1}-s)ds}.$$

It is however important to remember that when $\Phi_k \neq 1$, the volatility is no longer time-homogenous and it is therefore convenient to, approximately, preserve time-homogeneity by constraint $c_1 < \Phi_k < c_2$ where typical values for $c_1$ and $c_2$ is $0.85 - 0.99$ and $1.01 - 1.15$, respectively.

# References


[1] Bingham, N.H., Kiesel, R., 2004, *Risk-Neutral Valuation-Pricing and Hedging of Financial Derivatives.* Second Edition (London: Springer-Verlag).

[2] Björk, T., 2004, *Arbitrage Theory in Continuous Time.* Second Edition (Oxford: Oxford University Press).

[3] Blackham, J., 2004, *Sparse Grid Solutions to the LIBOR Market Model.* Master Thesis (Oxford: University of Oxford).

[4] Brace, A., Gaterek, D. and Musiela, M., 1997, *The Market Model of Interest Rate Dynamics.* Mathematical Finance, 7, 127-154.

[5] Brigo, D., Mercurio, F., 2006, *Interest Rate Models-Theory and Practice.* Second Edition (Berlin: Springer-Verlag).

[6] Brigo, D., Mercurio, F. and Morini, M., 2005, *The LIBOR Model Dynamics: Approximations, Calibraion and Diognastics.* European Journal of Operational Research, 163, 30-51.

[7] Fries, C., 2007, *Mathematical Finance: Theory, Modeling, Implementation.*(Hoboken: John Wiley & Sons).

[8] Grubišić, I., 2002, *Interest Rate Theory: The BGM Model.* Master Thesis (Leiden: Leiden University).

[9] Jamshidian, F., 1997, *LIBOR and Swap Market Models and Measures.* Finance and Stochastics, 1, 293-330.

[10] Kajsajuntti, L., 2004, *Pricing of Interest Rate Derivatives with the LIBOR Market Model.* Master Thesis (Stockholm: Department of Numerical Analysis and Computer Science, Royal Institute of Technology).

[11] Miltersen, K., Sandmann, K. and Sondermann, D., 1997, *Closed Form Solutions for Term Structure Derivatives with Log-normal Interest Rates.* Journal of Finance, 52(1), 409-430.

[12] Musiela, M., Rutkowski, M., 1997, *Continuous-time term structure models: Forward Measure Approach.* Finance and Stochastics, 1, 261-291.

[13] Packham, N., 2005, *Correlation Parameterization and Calibration for the LIBOR Market Model.* Master Thesis (Frankfurt: The Frankfurt School of Finance & Management).

[14] Pietersz, R., 2003, *LIBOR Market Model.* Master Thesis (Leiden: Leiden University).

[15] Rebonato, R., 1998, *Interest-Rate Option Models.* Second Edition (Chichester: John Wiley & Sons).

[16] Rebonato, R., 2002, *Modern Pricing of Interest-Rate Derivatives: The LIBOR Market Model and Beyond.* (Princeton: Princeton University Press).

[17] Rebonato, R., 2004, *Volatility and Correlation.* Second Edition (Chichester: John Wiley & Sons).

[18] Rutkowski, M., 2005, *Market Models of LIBORS and Swap Rates.* Working Paper (Sydney: School of Mathematics, The University of New South Wales).

[19] Salminen, T., 2007, *Calibration of a Forward Rate Market Model.* Master Thesis (Helsinki: Department of Engineering Physics and Mathematics, Helsinki University of Technology).

[20] Šelić, N., 2006, *The LIBOR Market Model.* Master Thesis (Johannesburg: school of Computational and Applied Mathematics, University of the Witwatersrand).

[21] Svoboda-Greewood, S., 2007, *Volatility Specifications in the LIBOR market Model.* Phd Dissertation (Oxford: University of Oxford).