ONE-FACTOR INTEREST RATE MODELS: ANALYTIC SOLUTIONS AND APPROXIMATIONS

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ONE-FACTOR INTEREST RATE MODELS: ANALYTIC SOLUTIONS AND APPROXIMATIONS

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Abstract

ONE-FACTOR INTEREST RATE MODELS:
ANALYTIC EXPRESSIONS AND APPROXIMATIONS

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The uncertainty attached to future movements of interest rates is an essential part of the Financial Decision Theory and requires an awareness of the stochastic movement of these rates. Several approaches have been proposed for modeling the one-factor short rate models where some lead to arbitrage-free term structures. However, no definite consensus has been reached with regard to the best approach for interest rate modeling. In this work, we briefly examine the existing one-factor interest rate models and estimate the parameters of Vasiček and Hull-White (Extended Vasiček) Models by using Turkey's term structure. Moreover, a trinomial interest rate tree is constructed to represent the evolution of Turkey's zero coupon rates.

Keywords: one-factor short rate models, arbitrage, term structure, trinomial interest rate tree.

Öz

TEK FAKTÖRLÜ FAİZ HADDİ MODELLEMESİ: ANALİTİK CÖZÜMLER VE YAKLAŞIMLAR

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Faiz haddinin gelecekteki haraketleri ile ilgili belirsizlik Finansal Karar Kuramının önemli bir parçasıdır ve faiz haddinin stokastik hareketini incelemeyi gerektirir. Tek faktörlü kısa dönem faiz haddi modelleri için literatürde bir çok yaklaşım önerilmiş ve bunların bazıları arbitrajı engelleyen kuponsuz tahvillerin vade yapılarının ortaya çıkmasını sağlamıştır. Fakat faiz haddi modellemesi için en iyi yaklaşımı bulma konusunda fikir birliği sağlanamamıştır. Bu çalışmada, var olan tek faktörlü kısa dönem faiz haddi modelleri incelenmiş ve Türkiye'nin vade yapısı kullanılarak Vasiček ve Hull-White (Genişletilmiş Vasiček) modellerinin parametre tahmini yapılmıştır. Ayrıca, Türkiye'nin kuponsuz tahvil faizini değerlendirmesini göstermek için üçlü faiz haddi ağacı oluşturulmuştur.

Anahtar Kelimeler: kısa dönem faiz haddi modelleri, arbitraj, vade yapısı, üçlü faiz haddi ağacı .

To my family

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Chapter 1

INTRODUCTION

The uncertainty attached to the future movements of interest rates is an important part of the theory of financial decision making. Most investors are risk averse and risk is linked in particular to interest rates.

It is thus important to understand the factors that drive interest rates and the models associated with it. The issue of pricing interest rate derivatives has been addressed by the financial literature in a number of different ways. One of the oldest approaches is based on modelling the evaluation of the instantaneous short interest rate. This is still quite popular for pricing interest rate derivatives and for risk management purposes, and represents the most commonly used type of dynamical stochastic model for interest rates. Moreover, in literature there are many models on the instantaneous short interest rate, but despite bewildering number of models, little is known about how these models compare in terms of their ability to capture the actual behavior of the short rate. Therefore, one of the aim of this study is to give information about these models and allow readers to compare them. We can separate these models into two category, which are Equilibrium models and No-arbitrage models. Vasicek (1977), Dothan (1978), Courtadon (1982), CIR (1985), Ho-Lee (1986), Black-Karasinski (1991), Hull-White (Extended Vasicek) (1993), Hull-White (Extended CIR)(1993), Ait-Sahalia (1996), Mercurio- Moraleda (2000), etc., is a list of one-factor short rate models investigated in the thesis. Many books contain analytic solutions to some of these models, though no specific book contains all solutions. In the work, we united various solutions and, in addition we derived the analytic expression of the Courtadon Model. We solve these models analytically by using stochastic differential methods. For further reading about mathematical methods, Kloedan-Platen (1992) is recommended. Then, we deal with parameter estimation of some of these models using Turkey's treasury zero coupon bond data. For estimating parameters we use different kinds of discretization methods and compare the results. Moreover, by this work we cope with the missing data problem. Indeed, we can observe the limited number of price of bonds which have different time to maturity since the Turkey's finance market is not deep. But, in parameter estimation we need the time series for the one day interest rate and so we develop a methodology for forming the time series data.

In the preliminaries, we give the fundamental definitions and theorems in the mathematical and financial viewpoint of interest rate modelling. In the following chapter, we present these various types of instantaneous spot interest rate models as we mentioned above and their analytic solutions. In the third chapter, we introduce the discretization methods of some of the models mentioned in the previous chapter and parameter estimation of some of these models is done by regressing the interest rate on its first lag. Also after estimating parameters we predict the next day's interest rate by using Monte Carlo Method. In the last chapter, we mention about interest rate trees and we form interest rate tree of the Zero Coupon Bond of Turkey by using Hull-White (Extended Vasiček) Model.

Chapter 2

PRELIMINARIES

First of all, we want to introduce the terminologies and definitions commonly used in modelling the stochastic behaviors of interest rates.

Definition 2.1. A ***Zero Coupon Bond*** with maturity date T , is a contract which guarantees the holder 1 unit of money to be paid at the maturity date T. The price at time t of a bond with maturity date T is denoted by $P(t,T)$.

Definition 2.2. ***Principal Value*** is the stated face value of the bond which is paid at the time of maturity.

Also there exists coupon bonds, which give the owner a payment stream during the life of the bonds. These instruments have the common property, that they provide the owner with a deterministic cash flow, and for this reason bonds are also known as *Fixed Income Instruments.* In order to guarantee the existence of sufficiently rich and regular bond markets, we should make the following assumption.

Assumption 2.1. We assume three conditions:

- There exists a frictionless (no tax, no transaction cost ...) market for bonds with each maturity.
- The relation $P(t,t) = 1$ holds for all t.
- For each fixed t, the bond price $P(t,T)$ is differentiable with respect to time of maturity T. (We suppose that T can be chosen as large as possible.)

The bond price $P(t,T)$ is a stochastic object with two variables, t and T, and for each outcome in the underlying sample space, the dependence upon these variables is very different. For a fixed value of t, $P(t,T)$ is a function of T. This function provides the prices, at the fixed time t, for bonds of all possible maturities. The graph of this function is called the *Term Structure at time t*. It is a real valued smooth graph, i.e., for each t, $P(t,T)$ is differentiable w.r.t T. The smoothness property is in fact a part of the above assumption. For a fixed maturity T, $P(t,T)$ is a stochastic process. This process gives the prices at different times, of the bond with fixed maturity T, and the trajectory is very irregular.

Definition 2.3. ***Arbitrage*** is the type of transaction in which an investor seeks to profit when the same asset are selling at two different prices. Indeed, the transaction involves buying the asset at lower price and selling it at the higher price. In literature it is called *free lunch*.

Note that in the assumption (2.1), $P(t,t) = 1$ should hold in order to avoid arbitrage. Suppose $P(t,t) < 1$, which means the price of a zero coupon bond at time t which expires on the same day is smaller than 1. Therefore, the arbitrageurs buy this bond and on the same day the bond expires and they get 1 unit of money.

In financial decision theory there are many kinds of interest rates and each has an important effect on the pricing of interest rate derivatives. Here are the commonly used ones in financial mathematics.

Definition 2.4. The main types of interest rates are as follows:

1. The ***simple forward rate*** for [S,T] contracted at t, henceforth referred to as the LIBOR forward rate, is defined as

$$L(t; S, T) = -\frac{P(t,T) - P(t,S)}{(T-S)P(t,T)}.$$

2. The ***continuously compounded forward rate*** for [S,T] contracted

at t is defined as

$$R(t; S, T) = -\frac{\log P(t,T) - \log P(t,S)}{T - S}.$$

3. The ***instantaneous forward rate*** with maturity T, contracted at t, is defined by

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}.$$

4. The ***instantaneous short rate*** at time t is defined by

$$r_t = f(t,t).$$

Basically, the short rate, r_t, is the rate that applies to an infinitesimally short period of time at time t. Indeed, short rate is actually a theoretical entity. It does not exist in real life and can not be observed directly.

Definition 2.5. The ***bank account process*** is defined by

$$B_t = \exp \int_0^t r_s \, ds,$$

i.e.,

$$\begin{cases} dB_t = r_t B_t \, dt \\ B_0 = 1. \end{cases}$$

Someone borrowing one unit of money at time t, until maturity at T, will have to pay back an amount $F(t,T)$ at time T, which is equivalent to an average interest rate R(t,T) given by the equality

$$F(t,T) \quad = \quad e^{(T-t)R(t,T)}$$

If we consider the future as *certain* , i.e., if we assume that all interest rates $(\mathrm{R(t,T)})_{t \leq T}$ are known, then, in an arbitrage free world, the function F must satisfy

$$\forall\ t < u < s : \quad F(t,s) \;=\; F(t,u)F(u,s)$$

From the relationship and the assumptions (which are the same as those on zero coupon bond price), if F is smooth there exists a function r_t (instantaneous short rate) such that

$$\forall\, t < T : \quad F(t,T) \;=\; \exp\,(\int_t^T r_s\, ds).$$

Therefore, the zero coupon bond price is as follows (since the position is inverse transaction of buying zero coupon bond):

$$P(t,T) \;=\; \exp\,(-\int_t^T r_s\, ds). \tag{2.0.1}$$

For an *uncertain* future, instantaneous rate can be considered as random process between times t and t+dt and then we can define the price of a zero coupon bond in an uncertain future. Our approach is based on stochastic calculus based on Brownian motion. We hereby give the the necessary mathematical frame of the approach. Before starting the definitions, we want to give a brief history of Brownian motion for motivation.

History of Brownian Motion

The history of Brownian Motion comes from 1828, when the Scottish botanist Robert Brown observed pollen particles in suspension under a microscope and observed that they were in constant irregular motion. By doing the same operations with particles of dust, he was able to rule out that the motion was due to *alive* pollen particles. In 1900, Bacheliar was the first person to model stock prices as a Brownian motion. Bachelier's model was his thesis and at that time it was thought that this topic was not worthy of study. In 1905, Einstein considered Brownian motion as a model of particles in suspension. He observed that, if the kinetic theory of fluids was right, then the molecules of water would move at random and so a small particle would receive a random number of impacts of random strength and from random directions in any short period of time. Such a bombardment would cause a sufficiently small particle to move in exactly the way described by Brown. In 1923, Nobert Wiener defined and constructed Brownian motion rigorously for the first time. The resulting stochastic process is often called the Wiener process in his honour [Sch03].

For various notions of Probability Theory used here, we refer to Körezlioğlu-Hayfavi(2001).

Definition 2.6. Consider a complete probability space $(\Omega, \mathcal{A}, P)$. An increasing family $\mathcal{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$ (we can take t$\in$ R$_+$ as well) is called a **filtration**. We put

$$\mathcal{F}_{t^+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}\ , \quad \mathcal{F}_{t^-} = \sigma\,(\bigcup_{s<t} \mathcal{F}_s), \quad \mathcal{F}_{0^-} = \mathcal{F}_0;$$

$\mathcal{F}$ is said to be *right continuous* if $\mathcal{F}_t = \mathcal{F}_{t^+}$ and *left continuous* if $\mathcal{F}_t = \mathcal{F}_{t^-}$. For $C \subset \mathcal{A}$, $\sigma(C)$ is the smallest sub-sigma-algebra of $\mathcal{A}$ containing C.

If for all t, $\mathcal{F}_t$ contains all $\mathcal{P}$-negligible sets, then $\mathcal{F}$ is called an *augmented filtration.* From now on we only consider augmented right continuous filtrations. In this case we say that the filtration $\mathcal{F}$ satisfies the *usual conditions.*

Definition 2.7. A ***Standard Brownian Motion*** is a continuous real valued stochastic process $W = \{W_t,\ t \geq 0\}$ with independent increments such that

$$W_0 = 0\ \ , \quad \text{and for} \quad 0 \leq s < t \quad W_t - W_s \sim N(0, t-s).^1$$

By a Brownian motion we always intend a standard Brownian motion. A d-dimensional Brownian motion is an $\mathbb{R}^d$ valued process whose components are mutually independent Brownian motions. Define

$$\mathcal{F}_t^W = \sigma\{W_s,\ \ s \leq t\};$$

[1] $W_t - W_s \sim N(0, t-s)$ means that $W_t - W_s$ has the normal distribution with mean zero and variance (t-s)

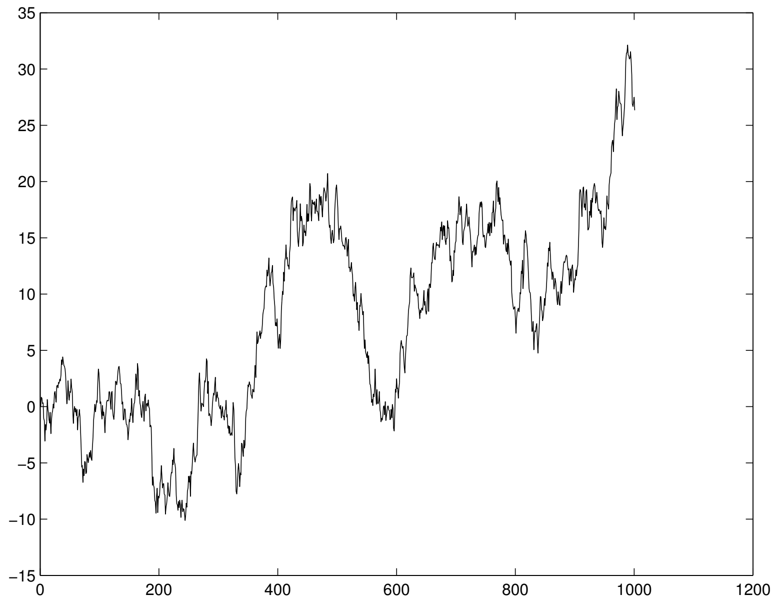

Figure 2.1: A sample path of standard Brownian motion

$\mathcal{F}^W$ is the raw filtration generated by W . We shall use the augmented filtration

$$\mathcal{F}_t = \sigma\{\mathcal{F}_t^W \cup \mathcal{N}\},$$

where $\mathcal{N}$ is the null subsets of $\mathcal{F}_T$.

Remark: The filtrations $\mathcal{F}^W$ and $\mathcal{F}$ are left continuous. The filtration $\mathcal{F}$ is right continuous, whereas $\mathcal{F}^W$ is not right continuous. Therefore, the filtration $\mathcal{F}$ satisfies the usual conditions.
From now on $\mathcal{F}^W = \{\mathcal{F}_t^W, \ t \geq 0\}$ will denote the augmented natural filtration of W.

Definition 2.8. A stochastic process on $(\Omega, \mathcal{A}, \mathcal{F}, P)$ is said to be adapted (to filtration $\mathcal{F}$) if $\forall t$ X_t is $\mathcal{F}_t$ measurable.

Given a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F} = \{\mathcal{F}_t, \ t \geq 0\}, P)$ an $\mathcal{F}$ - Brownian motion is a continuous real valued process $W = \{W_t, t \geq 0\}$ adapted to $\mathcal{F}$ with increments W_t - W_s independent of $\mathcal{F}_s$ for all $s < t$, such that $W_t - W_s \sim$ N(0,t-s). Obviously, a Brownian motion W is also $\mathcal{F}^W$ Brownian motion.

Definition 2.9. Let us consider a probability space $(\Omega, \mathcal{A}, P)$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ on this space. An adapted family $(M_t)_{t\geq 0}$ of integrable random variables, i.e., $E(|M_t|) < +\infty$ for any t is :

- a ***martingale*** if, for any $s \leq t,\ E(M_t|\mathcal{F}_s) = M_s$,
- a ***supermartingale*** if, for any $s \leq t,\ E(M_t|\mathcal{F}_s)\ \leq M_s$,
- a ***submartingale*** if, for any $s \leq t,\ E(M_t|\mathcal{F}_s)\ \geq M_s$.

Definition 2.10. A probability measure Q on $\mathcal{A}$ is absolutely continuous relative to P (i.e., $\forall$ A $\in \mathcal{A}$ P(A)=0 $\Rightarrow$ Q(A)=0) if and only if there exists a non-negative random variable Z on $(\Omega, \mathcal{A})$ such that

$$\forall\, A \in \mathcal{A}; \quad \mathrm{Q}(A) = \int_A Z(w)dP(w). \tag{2.0.2}$$

Here Z is called *density* of Q relative to P and denoted by $\frac{dQ}{dP}$, which is *Radon-Nikodym derivative* of Q relative to P.

Definition 2.11. ***(Girsanov Theorem)*** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a probability space and let $(B_t)_{0\leq t\leq T}$ be an $\mathcal{F}-$Brownian motion.
Let $(\theta_t)_{0\leq t\leq T}$ be an adapted measurable process satisfying $\int_0^t \theta_s^2\, ds < \infty$ a.s. and such that the process $(\mathrm{L}_t)_{0\leq t\leq T}$ defined by

$$L_t = \exp\,(\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds) \tag{2.0.3}$$

is a martingale. Then under the probability Q with density L_T relative to P, the process $(\mathrm{W}_t)_{0\leq t\leq T}$ defined by $\mathrm{W}_t = \mathrm{B}_t$ - $\int_0^t \theta_s\, ds$, is a $\mathcal{F}-$Brownian motion under Q.

Moreover for any nonnegative random variable X , we have $\mathrm{E}^Q(X)$=$\mathrm{E}(X\mathrm{L}_T)$ and if X is $\mathcal{F}_t$ measurable, then $\mathrm{E}^Q(X)$=$\mathrm{E}(X\mathrm{L}_t)$, setting L_t=$\mathrm{E}(\mathrm{L}_T|\mathcal{F}_t)$. Thus the random variable L_t is the density of Q restricted to $\mathcal{F}_t$ with respect to P [LL00] .

Definition 2.12. Any asset which has strictly positive prices for all t $\in [0, T]$ is called a ***numeraire***. We can use numeraires to denominate all prices in economy.

Definition 2.13. The prices Z_n of other assets denominated in terms of numeraire Z_1 are called ***relative prices*** and denoted by $\tilde{Z}_n = \frac{Z_n}{Z_1}$.

The value of bank account is assumed to earn a constant interest rate r and since it is strictly positive it can be served as numeraire. So the relative price of zero coupon bond price is denoted by ;

$$\begin{aligned} \tilde{P}(t,T) &= \frac{P(t,T)}{B(t,T)} \\ &= e^{-\int_0^t r_s ds} P(t,T). \end{aligned} \tag{2.0.4}$$

Also, $\tilde{P}(t, T)$ is the discounted bond price at time t with maturity T.

Definition 2.14. Let $(\Omega, \mathcal{A}, P)$ be a probability space and Q be any probability measure satisfies the followings:

- Q is equivalent to P, i.e., both measures have the same negligible sets;
- the relative price processes $\tilde{Z}_n$ are martingales under Q for all n, i.e., for s $\leq$ t we have $E^Q[\tilde{Z}_n(t)|\mathcal{F}_s] = \tilde{Z}_n(s)$.

Then the measures Q are called ***equivalent martingale measures***.

Theorem 2.1. ***(Unique Equivalent Martingale Measure)*** A continuous-time economy is arbitrage free and every security is attainable if for every choice of numeraire there exists a unique equivalent martingale measure.

We can paraphrase this absolutely important result as follows. Given a choice of numeraire (we always take as the bank account), we can find a unique probability measure such that for all T > 0 the relative price processes (discount prices in our example) are martingales.

Also note that for different kinds of numeraires there exists different unique equivalent martingale measure. On the other hand, in this work we only consider one numeraire which is the bank account process. So the following hypothesis is the most crucial one, since our approach is based on this hypothesis and, as a result, this leads to the formulation of the price of a zero coupon bond in an uncertain future.

MAIN SETTING: From now on we work on a filtered probability space $(\Omega, \mathcal{F}_\tau, \mathcal{F} = \{\mathcal{F}_t,\ \ t \in [0,\tau]\}, Q)$ where $\mathcal{F}$ is the natural filtration of a Brownian motion and τ is a large enough constant. Bonds maturity time will be taken on $[0,\tau]$. We define the probability measure Q by

$$dQ = L_\tau \, dP. \tag{2.0.5}$$

MAIN HYPOTHESIS: There exits a unique probability measure Q equivalent to P, under which, for all real valued T $\in$ [0,τ], the process $(\tilde{P}(t,T))_{0 \le t \le T}$ defined by

$$\tilde{P}(t,T) = e^{-\int_0^t r_s ds} P(t,T) \qquad \text{is a martingale.} \tag{2.0.6}$$

CONSEQUENCE: Since $P(t,T)$ is a martingale under Q and $P(T,T) = 1$ we have the following result :

$$\begin{aligned} \tilde{P}(t,T) &= E^Q(\tilde{P}(T,T)|\, \mathcal{F}_t) && (2.0.7)\\ &= E^Q(e^{-\int_0^T r_s ds} P(T,T)|\, \mathcal{F}_t) \\ &= E^Q(e^{-\int_0^T r_s ds}|\, \mathcal{F}_t) && (2.0.8)\end{aligned}$$

From equation (2.0.6), we can write $P(t,T)$ as follows:

$$\begin{aligned} \Rightarrow \ P(t,T) &= e^{\int_0^t r_s ds} \tilde{P}(t,T) \\ &= e^{\int_0^t r_s ds} E^Q\,(e^{-\int_0^T r_s ds}|\, \mathcal{F}_t) \\ &= E^Q\,(e^{-\int_t^T r_s\, ds}|\, \mathcal{F}_t). \qquad \square && (2.0.9)\end{aligned}$$

Therefore, the price of a zero coupon bond at time t with maturity T under uncertain future is;

$$P(t,T) = E^{Q}(e^{-\int_t^T r_s ds} | \mathcal{F}_t). \tag{2.0.10}$$

This equality represents that the prices $P(t,T)$ only depend on the behavior of the process $(r_s)_{0\leq s\leq T}$ under the probability Q.

Proposition 2.1. The price at time t of the zero coupon bond of maturity T≥t can be expressed as

$$P(t,T) = E\,(\,\exp\,(-\int_t^T r_s ds + \int_t^T \theta_s\, dB_s - \frac{1}{2}\int_t^T \theta_s^2\, ds) | \mathcal{F}_t). \tag{2.0.11}$$

where $(\theta_t)_{0\leq t\leq T}$ is an adapted measurable process such that $\int_0^T \theta_s^2\, ds < \infty$ P-a.s. and $L_t = \exp\,(\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds)$ is a positive P-martingale with $E(L_t) = 1$ for all t (as in (2.0.3)).

Proof. For any $Q-$integrable random variable X ;

$$E^{Q}(X|\mathcal{F}_t) = \frac{E(XL_T| \mathcal{F}_t)}{L_t}, \tag{2.0.12}$$

where L_t is defined in (2.0.3). Let $X = e^{-\int_t^T r_s ds}$. Then,

$$\begin{aligned} E^{Q}(e^{-\int_t^T r_s ds}| \mathcal{F}_t) &= \frac{E(e^{-\int_t^T r_s ds} L_T| \mathcal{F}_t)}{L_t} \\ &= \frac{E(e^{-\int_t^T r_s ds + \int_0^T \theta_s\, dB_s - \frac{1}{2}\int_0^T \theta_s^2\, ds}| \mathcal{F}_t)}{e^{\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds}} \\ &= E(\exp(-\int_t^T r_s ds + \int_t^T \theta_s dB_s - \frac{1}{2}\int_t^T \theta_s^2 ds)| \mathcal{F}_t). \quad \square \end{aligned}$$

Proposition 2.2. For each maturity T, there is an adapted process $(\sigma_t^T)_{0\leq t\leq T}$ such that on [0,T],

$$\frac{dP(t,T)}{P(t,T)} = (r_t - \sigma_t^T \theta_t)\, dt + \sigma_t^T dB_t. \tag{2.0.13}$$

Proof. Note that $\tilde{P}(t,T)_{0\leq t\leq T}$ is a martingale under Q (by main hypothesis). Then by using equality (2.0.12), for all $s < t$ we obtain as follows:

$$\begin{aligned} E(\tilde{P}(t,T)L_t|\mathcal{F}_s) &= E^Q(\tilde{P}(t,T)|\,\mathcal{F}_s)L_s \\ &= \tilde{P}(s,T)L_s, \end{aligned} \tag{2.0.14}$$

which means that $\tilde{P}(t,T)L_t$ is a martingale under P. Indeed, $(\tilde{P}(t,T)L_t)_{0\leq t\leq T}$ is a martingale under P whenever $\tilde{P}(t,T)_{0\leq t\leq T}$ is a martingale under Q. Moreover, $\tilde{P}(t,T)L_t > 0$ a.s., for all t $\in$ [0,T]. Then, there exists an adapted measurable process $(\theta_t^T)_{0\leq t\leq}$ such that $\int_0^T (\theta_t^T)^2 dt < \infty$ and

$$\tilde{P}(t,T)L_t = \tilde{P}(0,T)e^{\int_0^t \theta_s^T dB_s - \frac{1}{2}\int_0^t (\theta_s^T)^2 ds}.$$

Using the explicit expression of L$_t$ and the discount factor

$$P(t,T) = P(0,T)\exp\big(\int_0^t r_s ds + \int_0^t (\theta_s^T - \theta_s)dB_s - \frac{1}{2}\int_0^t (\theta_s^T)^2 - \theta_s^2 ds).$$

Applying the Ito formula with exponential function gives :

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= r_t dt + (\theta_t^T - \theta_t)dB_t - \frac{1}{2}((\theta_t^T)^2 - \theta_t^2)dt + \frac{1}{2}(\theta_t^T - \theta_t)^2 dt \\ &= (r_t + \theta_t^2 - \theta_t^T\theta_t)dt + (\theta_t^T - \theta_t)dB_t. \end{aligned} \tag{2.0.15}$$

which gives the equality (2.0.13) with $\sigma_t^T = \theta_t^T - \theta_t$ [LL00]. □

Remark Note that the riskless asset such as the bank account $dB_t = r_t B_t\, dt$ is related with formula (2.0.13) in the risk probability measure P. Moreover, the term $r_t - \sigma_t^T\theta_t$ corresponds intuitively to the average yield of the bond at time t and the term $-\sigma_t^T\theta_t$ is the difference between the average yield of the bond and riskless rate, thus θ_t can be interpreted as *risk premium*. From now on we shall denote the risk premium by λ. Under the probability Q , the process (W_t) is defined by $W_t = B_t - \int_0^t \theta_s ds$ (using Girsanov Theorem (Definition 2.11)), and then we get the following equation:

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma_t^T dW_t. \tag{2.0.16}$$

The important observation is that in order to specify completely the model, we have to provide λ. Indeed, the market price of risk λ connects the real-world measure to do risk neutral measure by the mathematical object $\frac{dP}{dQ}$. The way of moving one world to other world is characterization of our choice λ. On the other hand, if we just concern about the pricing of interest rate derivatives, we can directly model the short rate dynamics under the risk neutral measure Q, so that λ will be implicit in our dynamics.

Lemma 2.1. ***(Itô)*** Suppose we have a stochastic process X given by the stochastic differential equation $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$, and f be a twice continuously differentiable function. Then It$\hat{o}$ formula is :

$$df = (\,\mu(t,X_t)\frac{\partial f(t,X_t)}{\partial X} + \frac{1}{2}\sigma^2(t,X_t)\frac{\partial^2 f(t,X_t)}{\partial X^2})dt + \sigma(t,X_t)\frac{\partial f(t,X_t)}{\partial X}dW_t.$$

Likewise, if $(t,x) \mapsto f(t,x)$ is a function of class $\mathcal{C}^{1,2}$, It$\hat{o}$ formula becomes

$$df = (\frac{\partial f(t,X_t)}{\partial t} + \mu(t,X_t)\frac{\partial f(t,X_t)}{\partial X} + \frac{1}{2}\sigma^2(t,X_t)\frac{\partial^2 f(t,X_t)}{\partial X^2})dt + \sigma(t,X_t)\frac{\partial f(t,X_t)}{\partial X}dW_t.$$

Proposition 2.3. ***(Feynman-Kač Formula)*** Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t,X_t) + \mu(t,X_t)\frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2(t,X_t)\frac{\partial^2 F}{\partial X^2} - r_t F(t,X_t) = 0 \qquad (2.0.17)$$

$$F(T,X_t) = \Phi(X_T). \qquad (2.0.18)$$

Assume furthermore that the process $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{L}^2$, where X satisfies the SDE

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s \qquad (2.0.19)$$

under the probability measure Q. Then F has representation

$$F(t,X_t) = E^Q[e^{-\int_t^T r_s ds}\ \Phi(X_T)|\mathcal{F}_t]. \qquad (2.0.20)$$

Note that this type of partial differential equations appear in the study of pricing problems for financial derivatives.

Proof.

Consider the process

$$z_s = e^{-\int_t^s r_u du} F(s, X_s). \tag{2.0.21}$$

Then, using Itô Lemma and equation (2.0.19), we get

$$\begin{aligned} dz_s &= -r_s e^{-\int_t^s r_u\, du}\; F(s, X_s)\, ds + e^{-\int_t^s r_u\, du}\; dF(s, X_s) \\ &= -r_s e^{-\int_t^s r_u\, du}\; F(s, X_s)\, ds + e^{-\int_t^s r_u\, du}\; [F_s(s, X_s) ds \\ &\quad + F_X(s, X_s) dX_s + \frac{1}{2} F_{XX}(s, X_s)\; d < X, X >_s] \\ &= e^{-\int_t^s r_u\, du} [\, -r_s F(s, X_s) + F_s(s, X_s) + \mu F_X(s, X_s) \\ &\quad + \frac{1}{2} F_{XX}\, \sigma^2\,] ds + F_X\, \sigma\, dW_s. \end{aligned} \tag{2.0.22}$$

Since $F(t, X_t)$ satisfies the partial differential equation (2.0.17), dz_s turns to

$$dz_s \;=\; e^{-\int_t^s r_u\, du} \frac{\partial F}{\partial X}(s, X_s) \sigma\, dW_s.$$

Therefore, $(z_s)_{0 \le s \le T}$ is a Q-martingale, i.e.,

$$z_t = E^Q[z_T | \mathcal{F}_t]. \tag{2.0.23}$$

By using the equality (2.0.21) we can rewrite z_t as follows :

$$\begin{aligned} z_t &= e^{-\int_t^t r_u du} F(t, X_t) \\ &= F(t, X_t). \end{aligned}$$

Then by (2.0.23) we can write

$$F(t, X_t) = E^Q[e^{-\int_t^T r_s ds} \Phi(X_T) | \mathcal{F}_t]. \qquad \square \tag{2.0.24}$$

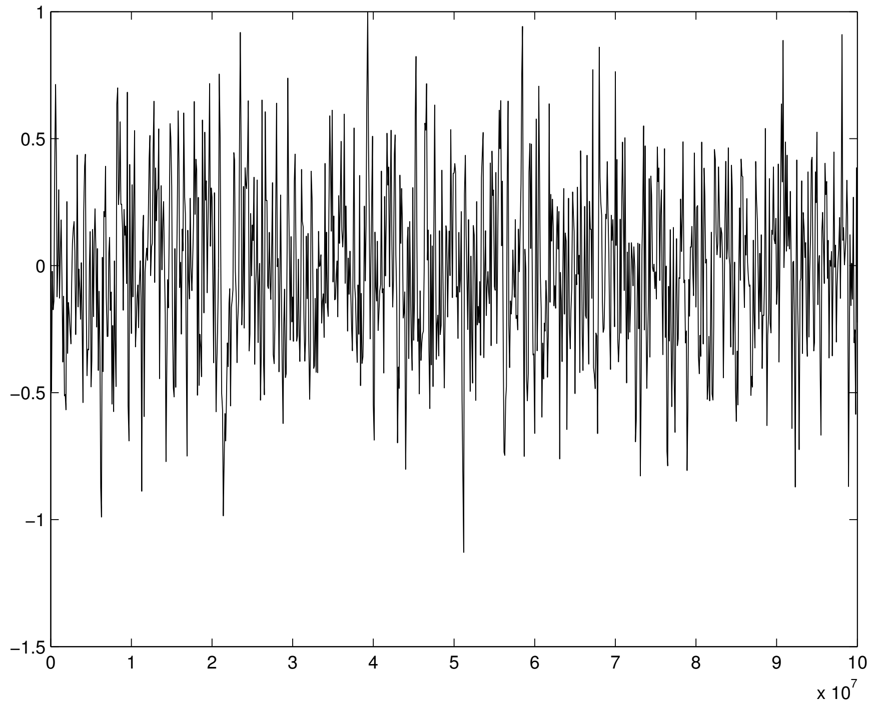

Figure 2.2: A sample path of Ornstein-Uhlenbeck Process for $c = 1,\ \ \sigma = 0.5$

Definition 2.15. An***Ornstein-Uhlenbeck Process*** is the unique solution of the following equation:

$$\begin{cases} dX_t = -cX_t\,dt + \sigma dW_t \\ X_0 = x \end{cases}$$

where $\sigma$and c be any constant.
It can be written explicitly as follows:

$$X_t \;\;=\;\; xe^{-ct} + \sigma e^{-ct}\int_0^t e^{cs}dW_s. \tag{2.0.25}$$

After reviewing financial and mathematical definitions, we can pass through the stochastic one-factor short rate interest rate models and their explicit solutions.

Chapter 3

ONE-FACTOR SHORT RATE MODELS

We consider a continuous trading market with no taxes, transactions costs, or short sale constraints. The uncertainty in this economy is represented by the complete filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}^{\mathcal{W}}{}_t)_t, Q)$ where Q is a risk neutral probability measure and $(\mathcal{F}_t)$ is the natural filtration of a Brownian motion W. The dynamics of the spot interest rate process is assumed to be represented by the following *stochastic differential equation (SDE):*

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t \tag{3.0.1}$$

with initial condition $r_0 = r_0$, where $\mu(.)$ and $\sigma^2(.)$ are respectively the drift and diffusion term of the process, and W_t is the standard Brownian Motion or Wiener process under the measure Q. In traditional spot interest rate models, $\mu(.)$ and $\sigma^2(.)$ are specified as simple parametric functions for pure simplicity and tractability. For any T ≥ 0 for existence and uniqueness of a solution in the interval $[0, T]$ of 3.0.1 we have three regularity conditions where K is a positive finite constant:

- $|\,\mu(t,x) - \mu(t,y)| + |\,\sigma(t,x) - \sigma(t,y)| \leq K|\,x - y|$ *(Lipschitz condition)*
- $|\,\mu(t,x)| + |\,\sigma(t,x)| \leq K(1 + |x|\,)$ *(growth condition)*

- $E(r_0^2) < \infty$.

These conditions guarantee that equation (3.0.1) has a unique solution satisfying $E(\sup |X_s|^2) < \infty$. From now on we suppose these conditions are satisfied. The first short rate models being proposed in the financial literature were time-homogeneous models whose diffusion coefficients are constant. The advantage of these models is analytic expression of bonds and bond options can be possibly expressed by using the dynamics of them. Moreover, they can be used to evaluate all interest rate contingent claims in a consistent way. However, the classic problem with them are their endogenous nature. If we have the initial zero coupon bond curve from the market, we wish our model to incorporate this curve, we need to force the model parameters to produce a model curve as close as possible to the market curve. Indeed, the initial term structure of rates need not to match the observed market data, no matter how the parameters are chosen. The main reason for this situation is, the current term structure of rates is an input rather than output. The other reason is that there are limited number of observed bond prices for a finite number of maturities and so the model parameters are not enough to reproduce satisfactorily a given term structure. As a result, they can not reproduce an initial yield curve satisfactorily. To improve this situation, exogenous term structures were introduced which are built by modifying the endogenous ones. The basic strategy is used to transform an endogenous model into exogenous model by using time-varying parameters. To illustrate, let us consider Vasiček model. Vasiček (1977) introduced the short rate dynamic as follows:

$$dr_t = \alpha(\beta - r_t)dt + \sigma\, dW_t,$$

where α, β and σ are any non negative constants. This model can be extended to

$$dr_t = \alpha(t)(\beta(t) - r_t) + \sigma dW_t,$$

where the function of $\beta(t)$ can be defined in terms of the market curve in such a way that the model reproduces exactly the curve itself at time 0. Ho-Lee (1986)

	Drift Term	Diffusion Term
EQUILIBRIUM MODELS		
Vasiček (1977)	$\beta(\alpha - r_t)$	σ
Dothan (1978)	βr_t	σr_t
Rendleman-Bartter (1980)	$\mu\, r_t$	σr_t
Courtadon (1982)	$\beta(\alpha - r_t)$	$\sigma\, r_t$
Constant Elasticity of Variance (CEV)	$\mu\, r_t$	$\sigma\, r_t^{\frac{\gamma}{2}}$
Marsh-Rosenfeld (1983)	$[\beta\, r_t^{-(1-\gamma)} + \alpha\, r_t]$	$\sigma\, r_t^{\gamma/2}$
Cox-Ingersoll-Ross (CIR)(1985)	$\beta(\alpha - r_t)$	$\sigma r_t^{\frac{1}{2}}$
Exponential Vasiček (EV)	$r_t[\eta - a \ln r_t]$	$\sigma\, r_t$
NO-ARBITRAGE MODELS		
Ho-Lee (1986)	$\theta(t)$	σ
Hull-White (Extended Vasiček) (1990)	$[\theta(t) - \alpha(t)\, r_t]$	$\sigma(t)$
Hull-White Extended CIR (1990)	$[\theta(t) - \alpha(t) r_t]$	$\sigma(t) r_t^{\frac{1}{2}}$
Black-Derman-Toy (1990)	$[\theta(t) + \frac{\sigma(t)'}{\sigma(t)} \ln r_t]$	$\sigma(t)$
Black-Karazinski (1991)	$[\alpha(t) - \beta(t) \ln r_t]$	σ

Table 3.1: One factor short-rate models

have been the first to propose an exogenous term structure model and their model was based on the assumption of a binomial tree governing the evaluation of the entire term structure rates [BM01]. Hull and White (1990), extension of CIR, etc., are the other exogenous models. Therefore, in this study we firstly examine the equilibrium models (which produce endogenous term structure) and then we examine the no-arbitrage models (which produce exogenous term structure). Here we give the well-known one-factor interest rate models that are discussed in literature. Also in all the models below we denote the initial value of r_t as r_0 which is supposed to be larger than 0.

3.1 One-Factor Equilibrium Models

These models are the first short rate models being proposed in the financial literature. As we said before, drift and diffusion terms contain constant parameters. Moreover, they produce an endogenous term structure (current term structure of rates is an output rather than an input of the model) of interest rates, in that initial term structure of rates need not to match the observed market data, no matter how the parameters are chosen.

3.1.1 Vasiček Model

(1977)

$$dr_t = \beta\,(\alpha - r_t)dt + \sigma dW_t,$$

where α, β, σ are non-negative constants [Vas77].

It is one of the earliest stochastic models of the short term interest rate. Vasiček assumed that the instantaneous spot rate under the real-world measure and risk-neutral measure as well evolves as an Ornstein-Uhlenbeck process with constant coefficients. So, the stochastic differential equation of short rate under the real probability P is;

$$dr_t = \beta\,(\alpha - r_t)dt + \sigma\, dB_t, \qquad (3.1.2)$$

where α, β, σ are non-negative constants and $B_t = W_t + \int_0^t \theta_s ds$. Although this model has many advantages, it has also shortcomings. The main advantage is that it has an explicit expressions. Moreover, r_t is normally distributed and the possibility of negative rates is indeed a major shortcoming of this model. In the model, the parameter α is the long run normal interest rate. Therefore, the model exhibits mean reversion, which means that if the interest rate is bigger than the long run mean (r$>\alpha$), then the coefficient $\beta(>0)$ makes the drift become negative so that the rate will be pulled down in the direction of r. Similarly, if the interest rate is smaller than the long run mean (r$>\alpha$), then the coefficient β (>0) makes the drift term become positive so that the rate will be pulled up in the direction of r. Therefore, the coefficient β is the speed of adjustment of the interest rate towards its long run level. There are also compelling economic arguments in favor of mean reversion. When the rates are high, the economy tends to slow down and borrowers require less funds. Furthermore, the rates pull back to its equilibrium value and the rates decline. On

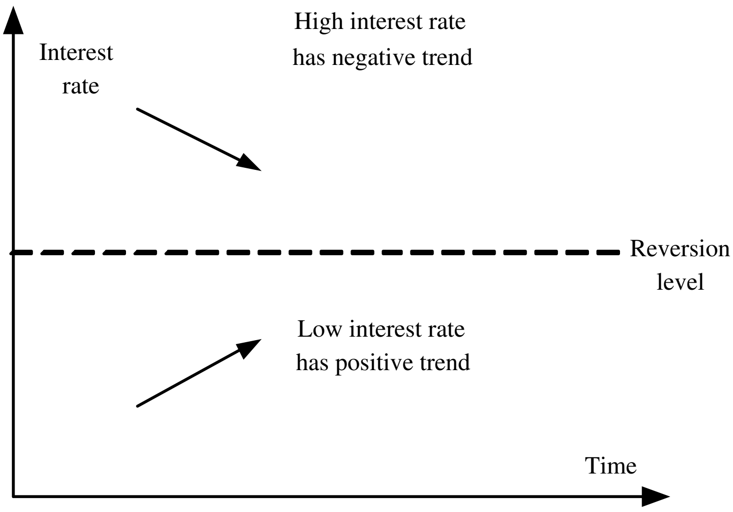

Figure 3.1: Mean reversion property

the contrary when the rates are low, there tends to be high demand for funds on the part of the borrowers and rates tend to increase. To understand better, we also draw the figure of the mean reversion property.

Analytic Expression:

Set $X_t := r_t - \alpha$.

Note that (X_t) is a solution of the stochastic differential equation
$dX_t = -\beta X_t dt + \sigma dW_t$ which is an *Ornstein-Uhlenbeck* process (Definition 2.13).
Set

$$Y_t := e^{\beta t} X_t; \tag{3.1.3}$$

herewith, we get :

$$\begin{aligned} dY_t &= \beta e^{\beta t} X_t dt + e^{\beta t} dX_t \\ &= \beta e^{\beta t} X_t dt + e^{\beta t}[-\beta X_t dt + \sigma dW_t] \\ &= e^{\beta t} \sigma dW_t. \end{aligned}$$

i.e.,

$$Y_t = Y_0 + \int_0^t e^{\beta s}\sigma \, dW_s. \tag{3.1.4}$$

By (3.1.3) we get

$$\begin{aligned} Y_0 &= e^0 X_0 \\ &= X_0. \end{aligned} \tag{3.1.5}$$

From (3.1.3), (3.1.4) and (3.1.5) we get:

$$\begin{aligned} e^{\beta t} X_t &= X_0 + \int_0^t e^{\beta s}\sigma \, dW_s \\ \rightarrow \ X_t &= e^{-\beta t}\,[X_0 + \int_0^t e^{\beta s}\sigma dW_s]. \end{aligned} \tag{3.1.6}$$

Since we set $X_t := r_t - \alpha$, then, r_t can be easily calculated as follows:

$$\begin{aligned} r_t &= e^{\beta t}(r_0 - \alpha) + e^{-\beta t}\sigma \int_0^t e^{\beta s} dW_s + \alpha \\ &= e^{-\beta t}(r_0 - \alpha) + \sigma \int_0^t e^{-\beta(t-s)} dW_s + \alpha \\ &= e^{-\beta t} r_0 + \alpha(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-s)} dW_s \\ &= e^{-\beta(t-u)} r_u + \alpha(1 - e^{-\beta(t-u)}) + \sigma \int_u^t e^{-\beta(t-s)} dW_s. \end{aligned} \tag{3.1.7}$$

Moreover, the conditional mean and the variance of r$_t$ are:

$$\forall \ u \leqslant t \qquad E^Q\{r_t|\, \mathcal{F}_u\} = e^{-\beta(t-u)} r_u + \alpha(1 - e^{-\beta(t-u)}), \tag{3.1.8}$$

$$\begin{aligned} \forall \ u \leqslant t \qquad Var\{r_t|\mathcal{F}_u\} &= E^Q\{(\sigma \int_u^t e^{-\beta(t-u)} dW_u)^2|\mathcal{F}_u\} \\ &= \frac{\sigma^2}{2\beta}[1 - e^{-2\beta(t-u)}]. \end{aligned} \tag{3.1.9}$$

3.1.2 Dothan Model

(1978)

$$dr_t = \beta r_t dt + \sigma r_t dW_t,$$

where β, σ are nonnegative constants [Dot78].

Indeed in the original paper of the Dothan(1978), he presented r_t by

$$dr_t = \sigma r_t \, dW_t \tag{3.1.10}$$

dynamic under the objective probability and assumed constant market price of the risk. Therefore, automatically the stochastic differential equation for short term interest rate turns into the equation under risk neutral probability that we mentioned above. Moreover, in Dothan (1978) model, the interest rate has a lognormal distribution which overcomes the drawback of negative values of the interest rate in the Vasiček Model.

Analytic Expression:

$$\begin{aligned}
r_t &= r_0 + \int_0^t \beta r_s ds + \int_0^t \sigma r_s dW_s \\
\ln r_t &= \ln r_0 + \int_0^t \frac{1}{r_s} dr_s + \frac{1}{2} \int_0^t \frac{-1}{r_s^2} d < r, r >_s \\
&= \ln r_0 + \int_0^t \frac{1}{r_s} \beta r_s ds + \int_0^t \frac{1}{r_s} \sigma r_s dW_s + \frac{1}{2} \int_0^t \frac{-1}{r_s^2} r_s^2 \sigma^2 ds \\
&= \ln r_0 + \int_0^t \beta ds + \int_0^t \sigma dW_s - \frac{1}{2} \int_0^t \sigma^2 ds \\
&= \ln r_0 + \beta t + \sigma W_t - \frac{1}{2} \sigma^2 t \qquad (3.1.11) \\
\Rightarrow \quad r_t &= r_0 \exp{(\beta t - \frac{1}{2} \sigma^2 t + \sigma W_t)}. \qquad (3.1.12)
\end{aligned}$$

$$\begin{aligned}
&= r_u \exp{\{\beta(t-u) + \sigma(W_t - W_u) - \frac{1}{2}\sigma^2(t-u)\}} \\
&= r_u \exp{\{(\beta - \frac{1}{2}\sigma^2)(t-u) + \sigma(W_t - W_u)\}}. \qquad (3.1.13)
\end{aligned}$$

Moreover, the conditional mean and the variance of r_t are :

$$\begin{aligned} \forall\ u \leqslant t: \qquad E\{r_t|\mathcal{F}_u\} &= r_u \exp\{(\beta - \frac{1}{2}\sigma^2)(t-u)\}E^Q\{e^{\sigma(W_t - W_u)}|\mathcal{F}_u\} \\ &= r_u \exp\{(\beta - \frac{1}{2}\sigma^2)(t-u)\}E^Q\{e^{\sigma W_{t-u}}|\mathcal{F}_0\} \\ &= r_u \exp\{(\beta - \frac{1}{2}\sigma^2)(t-u)\}exp\{\frac{1}{2}\sigma^2(t-u)\} \\ &= r_u \exp\{\beta(t-u)\}, \end{aligned} \tag{3.1.14}$$

$$\begin{aligned} \forall\ u \leqslant t: \qquad Var\{r_t|\mathcal{F}_u\} &= E^Q[\{r_u e^{(\beta-\frac{1}{2}\sigma^2)(t-u)}(e^{\sigma(W_t-W_u)} - 1)\}^2|\mathcal{F}_u] \\ &= r_u^2 e^{2\beta(t-u)}\{e^{-\sigma^2(t-u)}E^Q[e^{2\sigma W_{t-s}}|\mathcal{F}_0] \\ &\quad - 2e^{\frac{1}{2}\sigma^2(t-u)}E^Q[e^{\sigma W_{t-s}}|\mathcal{F}_0] + 1\} \\ &= r_u^2 e^{2\beta(t-u)}(e^{\sigma^2(t-u)} - 1). \end{aligned} \tag{3.1.15}$$

3.1.3 Rendleman-Bartter Model

(1980)

$$dr_t = \mu r_t dt + \sigma r_t dW_t,$$

where μ, σ are nonnegative constants [RM80].

When we look at roughly, we think that Dothan and Rendleman-Bartter have the same formulation. However, in the original paper of Dothan he aimed to present a valuation formula for default free bonds for a certain class of tastes when the rate follows a geometric Wiener process. And so he started with his work by the above formulation (3.1.10) under the objective probability. Since, we are dealing with the risk neutral valuation , we change directly this formulation to risk neutral framework.

Moreover, without doubt the analytic expression of this model is the same as that Dothan Model. Indeed, we will give the intuition of constructing this kind of model. Rendleman and Bartter assume that the short-term interest rate behaves like a stock price less than ideal. One important difference between the stock price and and interest rate is interest rate appeared to be pulled back to some long run average level over time (mean reversion) due to economic facts. Unlike Vasiček, they do not corporate with the mean reversion.

3.1.4 Courtadon Model

(1982)

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dW_t,$$

where α, σ, β are nonnegative constants [Cou82].

Courtadon also use the mean reversion property of interest rates. The main important difference between the Vasiček Model is that the diffusion term is not constant. Moreover, the analytic expression of this model can not be found in any book. Therefore, derivation of the solution is done by this work. Since it is not homogeneous stochastic differential equation, the steps get harder to solve this equation.

Analytic Expression:

Let us define

$$dX_t = -\beta X_t dt + \sigma X_t \, dW_t \tag{3.1.16}$$

Then,

$$X_t = X_0 - \int_0^t \beta X_s ds + \int_0^t \sigma X_s dW_s, \tag{3.1.17}$$

where X_0 is the initial value of X_t. By an informal application of the Itô Lemma, we can write:

$$\begin{aligned} \ln X_t &= \ln X_0 + \int_0^t \frac{1}{X_s} dX_s + \frac{1}{2}\int_0^t -\frac{1}{X_s^2} d < X, X >_s \\ &= \ln X_0 + \int_0^t \frac{1}{X_s}(-\beta X_s \, ds + \sigma X_s dW_s) - \frac{1}{2}\int_0^t \frac{1}{X_s^2}\sigma^2 X_s^2 \, ds \\ &= \ln X_0 - \int_0^t \beta ds + \int_0^t \sigma dW_s - \frac{1}{2}\int_0^t \sigma^2 \, ds \end{aligned} \tag{3.1.18}$$

From this we get:

$$X_t = X_0 \exp\{-(\beta + \frac{\sigma^2}{2})t + \sigma W_t\}. \tag{3.1.19}$$

By applying the Itô Lemma to this expression we see that this is a solution of (3.1.16). Since the coefficients of (3.1.16) satisfy the Lipschitz and linear growth conditions as in (3.0.1), (3.1.19) is the unique solution of (3.1.16).

Let

$$Y_t = \exp\{(\beta + \frac{\sigma^2}{2})\, t - \sigma W_t\} \tag{3.1.20}$$

and also define $Y_t = f(t, W_t)$, where $f(t,x) = \exp\{(\beta + \frac{\sigma^2}{2})t - \sigma x\}$. By the Itô Lemma, we know that:

$$f(t,W_t) = f(0,W_0) + \int_0^t f'_s(s,W_s)ds + \int_0^t f'_x(s,W_s)dW_s + \frac{1}{2}\int_0^t f'_{xx}(s,W_s)d < W, W >_s .$$

Since $Y_t = f(t, W_t)$ and $Y_0 = f(0, W_0) = 1$, we have

$$\begin{aligned} Y_t &= 1 + \int_0^t (\beta + \frac{\sigma^2}{2})Y_s\, ds - \int_0^t \sigma Y_s dW_s + \frac{1}{2}\int_0^t \sigma^2\, Y_s\, ds \\ &= 1 + \int_0^t (\beta + \sigma^2) Y_s ds - \int_0^t \sigma Y_s dW_s. \end{aligned}$$

i.e.,

$$dY_t = (\beta + \sigma^2)Y_t dt - \sigma Y_t dW_t \tag{3.1.21}$$

Let us set $Z_t := Y_t\, r_t$. Herewith, (3.1.22)

$$\begin{aligned} dZ_t &= Y_t dr_t + r_t dY_t + d < Y, r >_t \\ &= r_t[(\beta + \sigma^2)Y_t dt - \sigma Y_t dW_t] - \sigma^2 Y_t r_t dt \\ &\quad + Y_t[(\alpha\beta + \beta r_t)dt + \sigma r_t\, dW_t] \\ &= \alpha\beta Y_t dt. \end{aligned} \tag{3.1.23}$$

$\Rightarrow$

$$Z_t = Z_0 + \int_0^t \alpha\beta Y_s ds, \tag{3.1.24}$$

where Z_0 is the initial value of Z_t.
By (3.1.22), $\quad r_t = Z_t Y_t^{-1}$ and since $Y_0 = 1,\ Z_0 = r_0$, then

$$\begin{aligned} r_t &= Y_t^{-1}(r_0 + \int_0^t \alpha\beta Y_s ds) & (3.1.25)\\ &= r_0 \exp\{-(\beta + \frac{1}{2}\sigma^2)t + \sigma\, W_t\} \\ &\quad + \alpha\beta \int_0^t e^{-(\beta+\frac{1}{2}\sigma^2)(t-s)+\sigma(W_t-W_s)} ds. & (3.1.26)\end{aligned}$$

Finally we have

$$\begin{aligned} r_t &= r_u \exp\{-(\beta + \frac{1}{2}\sigma^2)(t-u) + \sigma(W_t - W_u)\} \\ &\quad + \alpha\beta \int_u^t e^{-(\beta+\frac{1}{2}\sigma^2)(t-s)+\sigma(W_t-W_s)} ds & (3.1.27)\end{aligned}$$

Conditional mean of r_t is given by

$$\begin{aligned} \forall\ u \leqslant t \qquad E^Q\{r_t|\,\mathcal{F}_u\} &= r_u e^{-(\beta+\frac{1}{2}\sigma^2)(t-u)} E^Q[e^{\sigma(W_t-W_u)}|\,\mathcal{F}_u] \\ &\quad + \alpha\beta E^Q[\int_u^t e^{-(\beta+\frac{1}{2}\sigma^2)(t-s)+\sigma W_{t-s}} ds|\mathcal{F}_u] \\ &= r_u e^{-(\beta+\frac{1}{2}\sigma^2)(t-u)} e^{\frac{1}{2}\sigma^2(t-u)} \\ &\quad + \alpha\beta \int_u^t E^Q[e^{-(\beta+\frac{1}{2}\sigma^2)(t-s)+\sigma W_{t-s}}|\,\mathcal{F}_s] ds \\ &= r_u e^{-\beta(t-u)} + \alpha\,(1 - e^{-\beta(t-u)}). & (3.1.28)\end{aligned}$$

3.1.5 Constant Elasticity of Variance Model

$$dr_t = \mu r_t dt + \sigma r_t^{\frac{\gamma}{2}} dW_t,$$

where μ, σ, γ are nonnegative constants [CYY99].

As it is known, the valuation of options has been one of the main issues in areas of financial mathematics and the Black-Scholes formula (1973) is the fundamental of option pricing with using constant volatility for stock price. Through Black-Scholes, constant volatility assumption is not very suitable in real cases. In view of this property, Cox (1975) and Cox-Ross (1976) studied the constant elasticity of variance (CEV) diffusion process which takes the form $dS_t = \mu S_t dt + \sigma S_t^{\frac{\gamma}{2}} dW_t$ where σ and γ are constants. Here μ is the expected rate of return for the stock price and variance rate of S_t is $\sigma^2 S^\gamma$. So the constant elasticity of variance (CEV) diffusion process are firstly used to model heteroscedasticity in returns to common stocks. Then, further studies are being done about CEV model which has same diffusion for interest rate processes. It is easily seen that the volatility of the model is $\sigma r_t^{\frac{\gamma}{2}-1}$.
When

$\gamma = 2$: model is just geometric Brownian motion and which become the same as Rendleman-Bartter Model (cf. Subsection 3.1.3);

$\gamma < 2$: volatility is a decreasing function of r_t;

$\gamma > 2$: volatility is an increasing function of r_t .

As a result, *Constant Elasticity of Variance Model* was a breakthrough in modelling volatility and interest rate process.

3.1.6 Marsh-Rosenfeld Model

(1983)

$$dr_t = (\beta\, r_t^{-(1-\gamma)} + \alpha\, r_t)\, dt + \sigma\, r_t^{\gamma/2}\, dW_t,$$

where $\alpha, \beta, \sigma, \gamma$ are nonnegative constants [MR83].

In the original paper of Marsh and Rosenfeld, they illustrated their methodology with a generalized case of the constant of elasticity of variance diffusion processes. If $\gamma = 1$, the Marsh-Rosenfeld Model turns to square root process with mean reverting drift term (which is considered by Cox-Ingersoll-Ross

Model,that we will mention after this model). If $\gamma = 0$, it turns to

$$dr_t = (\frac{\beta}{r_t} + \alpha r_t)dt + \sigma dW_t. \tag{3.1.29}$$

Moreover, when instantaneous short rate becomes small in (3.1.29), and $\beta > 0$, then the first term of the drift term dominates, and large positive changes are expected. On the contrary, if short rate is very large, the second term dominates, and the process behaves much like an Ornstein-Uhlenbeck process (cf. Definition 2.13).

With this assumption on the instantaneous short rate model (diffusion process) they fit the model to treasury bills by using maximum likelihood estimation [MR83].

3.1.7 Cox-Ingersoll-Ross Model

(1985)

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t^{1/2} dW_t,$$

where α, β, σ are nonnegative constants [CIR85] .
We can show that this model has a unique solution that is positive, but we do not have an explicit form for it.

Theorem 3.1. *Let r_t be the process satisfying*

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t^{1/2} dW_t.$$

Then its conditional expectation and the conditional variance are given by

$$E^Q\{r_t|\mathcal{F}_u\} = r_u e^{-\beta(t-u)} + \alpha(1 - e^{-\beta(t-u)}), \tag{3.1.30}$$

$$Var\{r_t|\mathcal{F}_u\} = r_u \frac{\sigma^2(e^{-\beta(t-u)} - e^{-2\beta(t-u)})}{\beta} + \frac{\alpha\sigma^2(1 - e^{-\beta(t-u)})^2}{2\beta}. \tag{3.1.31}$$

Proof.

If we take the integral of the stochastic differential of the model we have $\forall\, u \leq t$

$$r_t = r_u + \beta \int_u^t (\alpha - r_s) ds + \sigma \int_u^t r_s^{1/2} dW_s.$$

Applying Itô formula, we obtain

$$\begin{aligned} r_t^2 &= r_u^2 + 2\beta \int_u^t (\alpha - r_s) ds + 2\sigma \int_u^t r_s^{3/2} dW_s + \sigma^2 \int_u^t r_s ds \\ &= r_u^2 + (2\beta\alpha + \sigma^2) \int_u^t r_s ds - 2\beta \int_u^t r_s^2 ds + 2\sigma \int_u^t r_s^{3/2} dW_s. \quad (3.1.32) \end{aligned}$$

Note that r_t also can be written in terms of its initial value:

$$r_t = r_0 + \beta \int_0^t (\alpha - r_s) ds + \sigma \int_u^t r_s^{1/2} dW_s.$$

Then, the unconditional mean is

$$E^Q(r_t) = r_0 + \beta(\alpha t - \int_0^t E^Q(r_s) ds).$$

Solving the equation $\Phi(t) = r_0 + \beta(\alpha t - \int_0^t \Phi(s) ds)$, which can be transformed into the ordinary differential equation $\Phi(t)' + \beta\, \Phi(t) = \beta\alpha$, we get the unconditional mean:

$$E^Q(r_t) = \alpha + (r_0 - \alpha) e^{-\beta t} \quad (3.1.33)$$

Rearranging equality (3.1.33) we obtain:

$$E^Q\{r_t | \mathcal{F}_u\} = r_u e^{-\beta(t-u)} + \alpha(1 - e^{-\beta(t-u)}).$$

Similarly we firstly calculate the variance of r_t. Reorganizing equation (3.1.32) we obtain;

$$E^Q(r_t^2) = r_0^2 + (2\beta\alpha + \sigma^2) \int_0^t E^Q(r_s^2) ds - 2\beta \int_0^t E^Q(r_s^2) ds$$

Substituting the value of $E^Q(r_t)$ into the above equation and using the second moment, we can obtain the variance as follows:

$$Var(r_t) = \frac{\sigma^2}{\beta}(1-e^{-\alpha t})[r_0 e^{-\alpha t} + \frac{b}{2}(1-e^{-\alpha t})]. \qquad (3.1.34)$$

Rearranging equality (3.1.34) we obtain:

$$Var\{r_t|\mathcal{F}_u\} = r_u \frac{\sigma^2(e^{-\beta(t-u)} - e^{-2\beta(t-u)})}{\beta} + \frac{\alpha\sigma^2(1-e^{-\beta(t-u)})^2}{2\beta}.$$

In the original paper of Cox-Ingersoll and Ross, they aimed to have relationship between term structure of interest rates and yields on risk free securities that differ only in their time to maturity. The explanation of the term structure gives an extra information to predict the effect on the yield curve if we change the underlying variables.

The instantaneous short rate dynamics corresponds to *a continuous time first-order autoregressive process* where the randomly moving interest rate is elastically pulled toward a central location or long term value, α, which leads to mean reversion property. Also,

if $\sigma^2 > 2\alpha\beta$: r_t can reach zero,

if $\sigma^2 \leq 2\alpha\beta$: the upward drift is sufficiently large to make the origin inaccessible ,

In all cases : the singularity of the diffusion coefficient at the origin implies that an initially nonnegative interest rate can never be negative.

Moreover, by this model we do not face with negative interest rates. If the interest rate reaches zero, it can subsequently become positive. Cox-Ingersoll-Ross proposed a non-central chi-square distribution, $\mathcal{X}^2(2cr_s; 2q+2, 2u)$, with 2q+2 degrees of freedom and parameter of noncentrality 2u proportional to the current short rate where the probability density of the interest rate at time s, conditional on its value at the current time, t. It is given by the density function

$$f(r_s, s; r_t, t) = ce^{-u-v}(\frac{u}{v})^{\frac{q}{2}} I_q(2(uv)^{\frac{1}{2}}) \qquad (3.1.35)$$

where

$$\begin{aligned}
c &= \frac{2\beta}{\sigma^2(1 - e^{-\beta(s-t)})}, \\
u &= cr_t e^{-\beta(s-t)}, \\
v &= cr_s, \\
q &= \frac{2\beta\alpha}{\sigma^2} - 1,
\end{aligned}$$

and $I_q(.)$ is the modified *Bessel function* of the first kind of order q [CIR85].

3.1.8 Exponential-Vasiček Model

$$dr_t = r_t(\eta - a ln r_t)dt + \sigma r_t dW_t,$$

where a, σ, η are nonnegative constants [BM01-a].

Since the instantaneous short rate diffusion process is defined as the exponential of a process that is perfectly equivalent to that of Vasiček (1977), we can refer to this model as to the Exponential-Vasiček Model. Dothan (1978) was the first to think of the lognormal short rate model, but contrary to Dothan process, this r is always mean reverting. On the other hand, the Exponential-Vasiček model does not imply explicit formulas for either zero coupon bonds or options on them.

Analytic Expression:

Let y_t defined by;

$$dy_t = [\theta - ay_t]dt + \sigma dW_t, \tag{3.1.36}$$

where the initial value of y_t is y_0 and a, σ, θ are positive constants. The explicit solution of (3.1.36) is

$$
\begin{aligned}
y_t &= e^{-a(t-u)} y_u + \int_u^t \theta e^{-a(t-s)} ds + \int_u^t \sigma e^{-a(t-s)} dW_s \\
&= e^{-a(t-u)} y_u + \frac{\theta}{a}(1 - e^{-a(t-u)}) + \sigma \int_u^t e^{-a(t-s)} dW_s. \qquad (3.1.37)
\end{aligned}
$$

Set $r_t = \exp(y_t)$ where y_t satisfies the stochastic differential equation (3.1.36). By It$\hat{o}$ Lemma

$$
\begin{aligned}
r_t &= r_0 + \int_0^t r_s(\theta - a \ln r_s) ds + \int_0^t r_s\, \sigma dW_s + \frac{1}{2}\int_0^t \sigma^2\, ds \\
\Rightarrow \quad dr_t &= r_t[\theta + \frac{\sigma^2}{2} - a \ln r_s] dt + \sigma r_t\, dW_t. \qquad (3.1.38)
\end{aligned}
$$

Then the solution of r_t in the Exponential Vasiček model is

$$
\begin{aligned}
r_t &= e^{y_t} \\
r_t &= \exp[\ln r_u e^{-a(t-u)} + \frac{\theta}{a}(1 - e^{-a(t-u)}) + \sigma \int_u^t e^{-a(t-s)} dW_s]. \qquad (3.1.39)
\end{aligned}
$$

Moreover, the conditional mean and the variance of r_t are;

$$
\begin{aligned}
\forall\ u \leqslant t: \qquad E^Q\{r_t | \mathcal{F}_u\} &= \exp\{\ln r_u e^{-a(t-u)} + \frac{\theta}{a}(1 - e^{-a(t-u)}) \\
&\quad + E^Q\{\sigma \int_u^t e^{-a(t-s)} dW_s | \mathcal{F}_u\} \\
&= \exp\{\ln r_u e^{-a(t-u)} + \frac{\theta}{a}(1 - e^{-a(t-u)}) + \frac{\sigma^2}{4a}[1 - e^{-2a(t-s)}]\},
\end{aligned}
$$

$$
\begin{aligned}
\forall\ u \leqslant t: \qquad Var\{r_t | \mathcal{F}_u\} &= E^Q[\exp((\sigma \int_u^t e^{-a(t-u)} dW_u) - \frac{\sigma^2}{4a}[1 - e^{-2a(t-s)}])^2 | \mathcal{F}_u] \\
&= \exp\{2 \ln r_s e^{-a(t-u)} + 2\frac{\theta}{a}(1 - e^{-a(t-u)}) + \frac{\sigma^2}{a}(1 - e^{-2a(t-u)})\}.
\end{aligned}
$$

Also, the asymptotic unconditional mean and variance looks and behaves as follows:

$$\lim_{t\to\infty} E^Q(r_t) = \exp(\frac{\theta}{a} + \frac{\sigma^2}{4a}) \tag{3.1.40}$$

$$\lim_{t\to\infty} = \exp(\frac{2\theta}{a} + \frac{\sigma^2}{2a})(\exp(\frac{\sigma^2}{2a}) - 1) \tag{3.1.41}$$

3.2 No-Arbitrage Models

The disadvantage of equilibrium models presented in the previous section is that they do not automatically fit today's term structure. By modifying parameters, such as taking parameters time dependent we can fit one or two bonds exactly, but fitting the whole bonds which have different maturities leads to significant errors. Moreover, another disadvantage is there exist errors in option pricing using the Equilibrium Models.

To cope with the fitting problems to observed data, *No-Arbitrage Models* are designed to be exactly consistent with today's term structure of interest rates. As we know, in an equilibrium model such as Vasiček, today's term structure is an output, which means that by using this model we estimate the value of the short rate for distinct time to maturity dates. However, the short rates for some time to maturity dates have been determined. Therefore, there is a confliction between the model and observed actual rate. Therefore, these kinds of models leads to arbitrage. On the other hand, in a no-arbitrage model, today's term structure is an input, i.e. while constructing the model we take the observed actual rates and estimate the unobserved rates. Moreover,in the *No-Arbitrage Models* the drift term of the instantaneous short rate diffusion is in general a function of time, i.e. it depends on time. This is because the shape of the initial zero coupon curve governs the average path taken by the short rate in the future in a no-arbitrage model. If the term structure is upward sloping, then r will increase in a risk neutral world. If the term structure is downward sloping, then r will decrease in a risk neutral world, and if the term structure first increase and then decrease, the expected term structure has positive slope

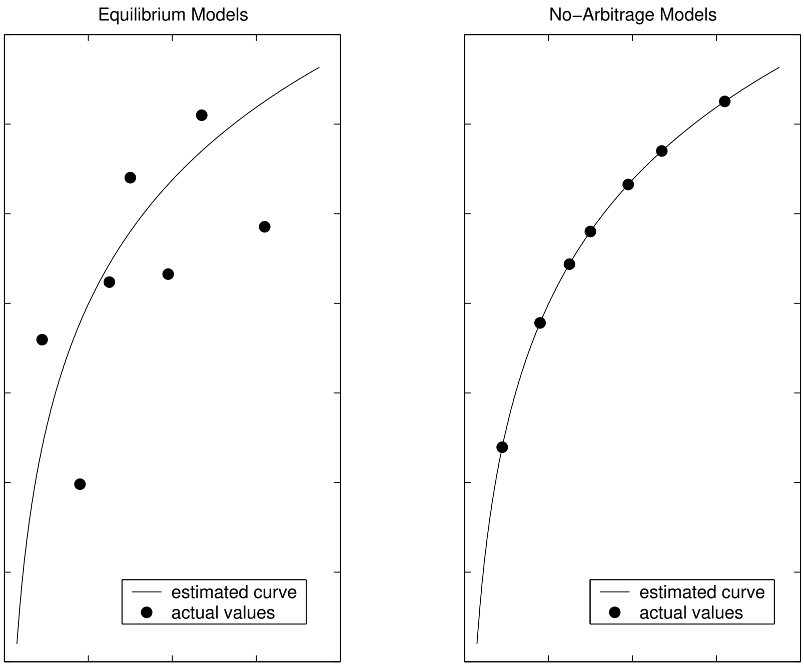

Figure 3.2: Estimation of short rate with Equilibrium and No-Arbitrage models

initially and then a negative slope later.

We can transform an endogenous model (equilibrium model) into an exogenous model (no-arbitrage model) by including a function of time in the drift term of the short rate. In the latter sections some examples will be given on these transformations and here are the description and analytic solution of some major no-arbitrage models.

3.2.1 Ho-Lee Model

(1986)

$$dr_t = \theta(t)dt + \sigma dW_t,$$

where θ is non-random function of t and σ is nonnegative constant [HL86].

Ho and Lee have been the first to propose no-arbitrage model of the term structure as opposed to models that endogenously produce the current term structure of rates. The main advantage of their approach is that it enables us to utilize the full information of the term structure to price contingent claims. They presented the model in the form of a binomial tree of bond prices with two parameters; which are the short rate standard deviation and the market price of the risk of the short rate in discrete time. Under the risk neutral measure, the continuous time model turns to above stochastic differential equation. The variable $\theta(t)$ defines the average direction that short rate moves at time t. An interesting fact is that when the model is used to price interest rate derivatives, the parameter that concerns the market price of risk proves to be irrelevant. The main drawback of this model is that Ho and Lee did not consider the mean reversion property as Vasiček.

Analytic Expression:

Firstly, let us find the solution of this equation and then we will calculate $\theta(t)$ analytically by using bond prices:

$$
\begin{aligned}
r_t &= r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s && (3.2.42)\\
&= r_0 + \int_0^t \theta(s)ds + \sigma W_t && (3.2.43)\\
&= r_u + \int_u^t \theta(s)ds + \sigma[W_t - W_u]. && (3.2.44)
\end{aligned}
$$

To fit this model to the initial term structure of interest rates, we have to calculate the zero coupon bond prices in terms of $\theta(t)$. We have shown that the price of a zero coupon bond at present with maturity T is

$$P(0,T) = E^Q(e^{-\int_0^T r_s ds}).$$

Let

$$y(t) := \int_0^t r_s\, ds.$$

Then we can express $P(0, T)$ as follows:

$$P(0, T) = E^Q(e^{-y(T)}),$$

where

$$\begin{aligned} y(t) &= \int_0^t r_0 \, ds + \int_0^t \int_0^s \theta(u) \, duds + \int_0^t \int_0^s \sigma \, dW_u ds \\ &= r_0 t + \int_0^t \int_u^t \theta(u) \, dsdu + \int_0^t \int_u^t \sigma \, dsdW_u. \end{aligned}$$

Simplifying the terms we obtain;

$$y(t) = r_0 \, t + \int_0^t \theta(u)(t-u)du + \int_0^t \sigma(t-u)dW_u.$$

From the short rate dynamics of Ho-Lee it is easily seen that the interest rates are normal. So $y(t) = \int_0^t r_s \, ds$ which is just sum is also normally distributed. Then $y(t)$ has mean with

$$m(t) = r_0 t + \int_0^t \theta(s)(t-s)ds,$$

and variance

$$v(t) = \int_0^t \sigma^2 (t-s)^2 ds = \frac{1}{3}\sigma^2 t^3.$$

From the expression of the Laplace transform of Gaussian:

$$\begin{aligned} P(0, T) &= E^Q(e^{-y(T)}) \\ &= \exp\left(-E^Q(y(T))\right) + \frac{1}{2} Var(y(T))) \qquad (3.2.45) \\ &= \exp\left(-m(T)\right) + \frac{1}{2} v(T)) \\ &= \exp\left[-r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3\right]. \qquad (3.2.46) \end{aligned}$$

Taking the logarithms and differentiating twice with respect to T yields

$$\theta(T) = -\frac{\partial^2}{\partial T^2} logP(0,T) + \sigma^2 T.$$

Also note that from the definition 2.4, the instantaneous forward rate is

$$f(0,T) = -\frac{\partial logP(0,T)}{\partial T},$$

then θ can also be expressed in terms of forward rates as follows:

$$\theta(T) = \frac{\partial f(0,T)}{\partial T} + \sigma^2 T.$$

3.2.2 Hull-White Model (Extended Vasiček)

(1990)

$$dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dW_t,$$

where $\alpha(t), \theta(t), \sigma(t)$ are nonrandom functions of t [HW90]. This one factor model is just an extension of Vasiček (1977). The volatility of the instantaneous short rate can be a function of time. In the previous model, Ho-Lee do not deal with mean reversion property, and by this model Hull-White corporate the mean reversion property.

Analytic Expression:

Set

$$K(t) := \int_0^t \alpha(s)ds,$$

$$\begin{aligned} Then \quad d\,(e^{K(t)}r_t) &= e^{K(t)}(K'(t)r_t dt + dr_t) \\ &= e^{K(t)}[\alpha(t)r_t\,dt + (\theta(t) - \alpha(t)r_t)\,dt + \sigma(t)\,dW_t] \\ &= e^{K(t)}(\theta(t)\,dt + \sigma(t)\,dW_t). \end{aligned} \tag{3.2.47}$$

Integrating (3.2.47), we get

$$\begin{aligned}
e^{K(t)}r_t &= r_0 + \int_0^t e^{K(s)}\theta(s)ds + \int_0^t e^{K(s)}\sigma(s)dW_s \\
\Rightarrow \quad &= e^{-K(t)}[r_0 + \int_0^t e^{K(s)}\theta(s)ds + \int_0^t e^{K(s)}\sigma(s)dW_s] \\
&= e^{-K(t)}r_0 + \int_0^t e^{-(K(t)-K(s))}\theta(s)ds + \int_0^t e^{-(K(t)-K(s))}\sigma(s)dW_s \\
&= e^{-(K(t)-K(u))}r_u + \int_u^t e^{-(K(t)-K(s))}\theta(s)ds \\
&\quad + \int_u^t e^{-(K(t)-K(s))}\sigma(s)dW_s. \qquad (3.2.48)
\end{aligned}$$

Usually $\alpha(t)$ and $\sigma(t)$ are supposed to be constant and the solution of r_t turns to

$$\begin{aligned}
r_t &= e^{-\alpha t}r_0 + \int_0^t e^{-\alpha(t-s)}\theta(s)ds + \int_0^t e^{-\alpha(t-s)}\sigma dW_s \qquad (3.2.49) \\
&= e^{-\alpha(t-u)}r_u + \int_u^t e^{-\alpha(t-s)}\theta(s)ds + \int_u^t e^{-\alpha(t-s)}\sigma dW_s \qquad (3.2.50)
\end{aligned}$$

The conditional mean of r_t is

$$\forall\ u \leqslant t: \quad E^Q\{r_t|\mathcal{F}_u\} = r_u e^{-\alpha(t-u)} + E^Q[\int_u^t e^{-\alpha(t-s)}\theta(s)ds|\mathcal{F}_u]. \qquad (3.2.51)$$

To fit this model to the initial term structure of interest rates , we have to calculate the zero coupon bond prices in terms of $\theta(t)$. We have shown that the price of a zero coupon bond at present with maturity T is

$$P(0,T) = E^Q(e^{-\int_0 T r_s ds}).$$

Let

$$y(t) = \int_0^t r_s ds.$$

Then we can express P(0,T) as follows:

$$P(0,T) = E^Q(e^{-y(T)}).$$

By using equation (3.2.49), $y(t)$ turns out to be;

$$\begin{aligned} y(t) &= \int_0^t e^{-\alpha s} r_0 ds + \int_0^t \int_0^s e^{-\alpha(s-u)} \theta(u) du\, ds + \int_0^t \int_0^s \sigma e^{-\alpha(s-u)} dW_u\, ds \\ &= -\frac{r_0}{\alpha}(e^{-\alpha t} - 1) + \int_0^t \int_u^t e^{-\alpha(s-u)} \theta(u) ds\, du \\ &+ \int_0^t \int_u^t \sigma e^{-\alpha(s-u)} ds\, dW_u. \end{aligned}$$

Simplifying the terms we obtain;

$$\begin{aligned} y(t) &= -\frac{r_0}{\alpha}(e^{-\alpha t} - 1) + e^{-\alpha s} \int_0^t e^{\alpha u} \theta(u)(t-u)\, du \\ &+ e^{-\alpha s} \int_0^t e^{\alpha u} \sigma(t-u)\, dW_u. \end{aligned}$$

From the short rate dynamics of Hull-White it is easily seen that the interest rates are normal. So $y(t) = \int_0^t r_s ds$ which is a just sum of normally distributed random variables. Therefore, it is also normally distributed. Then, $y(t)$ has mean with

$$m(t) = -\frac{r_0}{\alpha}(e^{-\alpha t} - 1) + e^{-\alpha s} \int_0^t e^{\alpha u} \theta(u)(t-u)\, du,$$

and variance

$$v(t) = e^{-2\alpha s} \sigma^2 [-\frac{t^2}{2\alpha} - \frac{t}{2\alpha^2} + \frac{e^{2\alpha t}}{4\alpha^3} - \frac{1}{4\alpha^3}].$$

From the expression of the Laplace transform of Gaussian:

$$\begin{aligned} P(0,T) &= E^Q(e^{-y(T)}) \\ &= exp(-E^Q(y(T))) + \frac{1}{2} Var(y(T))) \qquad (3.2.52) \\ &= exp(-m(T)) + \frac{1}{2} v(T)) \\ &= exp\{-\frac{r_0}{\alpha} e^{-\alpha T} + \frac{r_0}{\alpha} + e^{-\alpha s} \int_0^T e^{\alpha u} \theta(u)(T-u) du \\ &+ \frac{1}{2} e^{-2\alpha s} \sigma^2 [-\frac{T^2}{2\alpha} - \frac{T}{2\alpha^2} + \frac{e^{2\alpha T}}{4\alpha^3} - \frac{1}{4\alpha^3}]\}. \qquad (3.2.53) \end{aligned}$$

Taking the logarithms and differentiating twice with respect to T and using the equality for instantaneous forward rate, $f(0,T) = -\frac{\partial log P(0,T)}{\partial T}$, then θ can be expressed in terms of forward rates as follows:

$$\theta(T) = \frac{\partial f(0,T)}{\partial T} + \alpha f(0,T) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha T}). \qquad (3.2.54)$$

3.2.3 Hull-White Model (Extended CIR)

(1990)

$$dr_t = (\theta(t) - \alpha(t) r_t) dt + \sigma(t)\sqrt{r_t} dW_t,$$

where $\alpha(t), \theta(t), \sigma(t)$ are nonrandom functions of t [HW90].

CIR Model is extended by Hull and White by introducing time-varying parameter, θ . Indeed, it is the time varying mean. Also, matching the model and the market term structures of rates at the current time is equivalent to solving a system with an infinite number of equations, one for each possible maturity. Moreover, in practice we take the diffusion term constant in order to make our calculations simple.

3.2.4 Black-Derman-Toy Model

(1990)

$$d \ln r_t = \theta(t)\, dt + \sigma\, dW_t,$$

where $\theta(t)$ is a non-random function of t and σ is nonnegative constant [BDT90].

In their original paper, they proposed a discrete time approach of interest rate. The continuous time equivalent of their model was as follows:

$$d \ln r_t = (\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r_t)\, dt + \sigma(t)\, dW_t, \qquad (3.2.55)$$

where the deterministic function $\sigma(t)$ is chosen to make the consistent with the term structure of spot rate volatilities. Also, they presented this short rate model to valuate the treasury bond options. This model is one of the best known and the most widely used models. Capability to price exactly an arbitrary set of zero coupon bonds, ease of calibration and the log normal short rate are the main features of this model.

Analytic Expression:

First of all we solve $\ln r_t$ and then, by taking logarithm of both sides, we obtain the exact solution of the instantaneous short rate:

$$\begin{aligned}
\ln r_t &= \ln r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma\, dW_s \\
&= \ln r_0 + \int_0^t \theta(s)\, ds + \sigma\, W_t \\
\rightarrow r_t &= r_0 \exp\,[\int_0^t \theta(s)\, ds + \sigma\, W_t] \\
r_t &= r_u \exp\,[\int_u^t \theta(s)\, ds + \sigma(W_t - W_u)]. \qquad (3.2.56)
\end{aligned}$$

3.2.5 Black-Karasinski Model

(1991)

$$d \ln r_t = (\alpha(t) - \beta(t) \ln r_t)dt + \sigma dW_t,$$

where $\alpha(t), \beta(t)$ are non-random functions of t and σ is nonnegative constant [BK91].

Black-Karasinski Models is actually a generalization of the Black-Derman-Toy (1990) model. This model has been used to present bond prices, bond yields and related options. It is the single factor since they assumed the source of all uncertainty is short term interest rate model. In their paper, they told that they could use the other models such as ”square root process” as CIR (1985) used for overcoming the negative interest rate, but the nominal interest rate can not fall below zero as long as people hold cash; it can become stuck at zero for long periods, however, as when prices fall persistently and substantially. None of the models except lognormal are convenient to these features. Lognormal models keep the rate away from zero entirely, while some square root models make zero. On the other hand, they used more general lognormal model, since they wanted to allow the local process to change over time. Also, since the market formulas for caps are based on the assumption of lognormal rates, it seemed reasonable to choose the same distribution for the instantaneous short rate processes.

Note that good fitting quality of the model to market data, has made the model quite popular among practitioners and financial engineers. Here is the analytic solution of the model.

Analytic Expression:

Set

$$K(t) := \int_0^t \beta(s)ds.$$

Let $Y_t := \ln r_t$; herewith,

$$\begin{aligned} d(e^{K(t)}Y_t) &= e^{K(t)}K'(t)Y_t dt + e^{K(t)}dY_t \\ &= e^{K(t)}\beta(t)Y_t dt + e^{K(t)}(\alpha(t) - \beta(t)Y_t)dt + \sigma dW_t) \\ &= e^{K(t)}(\alpha(t)dt + \sigma dW_t). \end{aligned} \tag{3.2.57}$$

Integrating both sides gives

$$e^{K(t)}Y_t = Y_0 + \int_0^t e^{K(s)}\alpha(s)ds + \int_0^t \sigma e^{K(s)}dW_s \tag{3.2.58}$$

$$\begin{aligned} \Rightarrow \qquad Y_t &= e^{-K(t)}Y_0 + \int_0^t e^{-(K(t)-K(s))}\alpha(s)ds \\ &\quad + \int_0^t \sigma e^{-(K(t)-K(s))}dW_s \qquad (3.2.59) \\ &= e^{-(K(t)-K(u)}Y_u + \int_u^t e^{-(K(t)-K(s))}\alpha(s)ds \\ &\quad + \int_u^t \sigma e^{-(K(t)-K(s))}dW_s. \qquad (3.2.60) \end{aligned}$$

Since $Y_t = \ln r_t$, we can write the following equality:

$$\ln r_t = e^{-(K(t)-K(u))} lnr_u + \int_u^t e^{-(K(t)-K(s))}\alpha(s)ds + \int_u^t \sigma e^{-(K(t)-K(s))}dW_s$$

$$\Rightarrow \quad r_t = \exp\{\ln r_u\, e^{-(K(t)-K(u))} + \int_u^t e^{-(K(t)-K(s))}\alpha(s)ds + \int_u^t \sigma e^{-(K(t)-K(s))}dW_s\}. \quad (3.2.61)$$

3.3 Some Other Extended Models

These models are derived by extending the time homogeneous short rate models to models that can reproduce any observed yield curve. While looking at the literature about extended models, we can easily observed that Brigo and Mercurio worked on them. They were able to exactly fit any observed term structure of interest rates and derived analytic expressions for interest rate derivative. However, the drawback of this procedure is there should be restrictions on the parameters in order to satisfy positivity of interest rates. Moreover, the restrictions on the parameters of the models might worsen the calibration to caps or swaption prices [BM01-a]. Here are the examples of some extended models.

3.3.1 CIR ++ Model

(1998)

$$r_t = x_t + \varphi(t) \quad dx_t = \alpha(\mu - x_t)\,dt + \sigma\sqrt{x_t}\,dW_t,$$

where k, θ, σ are nonnegative constants [BM01-b].
This model is just extension of CIR (1985) and Extended CIR (1990). Brigo and Mercurio proposed this model because of the analytic dis-tractability of previous CIR models. Morover, this model can fit any initial term structure. This is done by

$$\varphi(t) = f^M(0,t) - f^{CIR}(0,t)$$

where f^M is the market forward rate and f^{CIR} is the forward rate derived by the model.

3.3.2 Mercurio-Moraleda Model

(2000)

$$dr_t = r_t(\eta(t) - (\lambda - \frac{\gamma}{1+\gamma t}) \ln r_t)dt + \sigma r_t \, dW_t,$$

where η is nonrandom function of t and γ, σ are nonnegative functions [BM01-a].

3.3.3 Extended Exponential Vasiček Model

(2001)

$$r_t = x_t + \varphi_t, \quad dx_t = x_t(\eta - a \, \ln x_t) \, dt + \sigma \, x_t \, dW_t,$$

where a, η, σ are nonnegative constants [BM01-a].
Remember the Exponential Vasiček model that the instantaneous short rate evolves as the exponential of an *Ornstein -Uhlenbeck* pocess y_t defined in (3.1.36). Moreover, $\varphi(t)$ is defined as follows:

$$\varphi(t) = f^M(0,t) - f^{EEV}(0,t)$$

where f^M is the market forward rate and f^{EEV} is the forward rate derived by the model.

Chapter 4

DISCRETIZATION

The aim of this chapter is to compare the results of discretization techniques while estimating the parameters of some one-factor short rate models. The methods that are discussed in this chapter are Euler and Milshtein approximations. The parameter estimation of some models are calculated by Euler scheme and we will predict the next day interest rate for zero coupon bond. The simplest approximation is the *Euler scheme*. Although it is easy to implement it is not always sufficiently accurate.

4.1 The Euler Scheme

First of all let us consider the process X satisfying the stochastic differential equation of the form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \tag{4.1.1}$$

with the initial value X_0 fixed. Define $\hat{X}$ which is the time discretized approximation to X. Approximation is on time grid $0 = t_0 < t_1 < \ldots < t_m$ where $\hat{X}_0 = X_0$ and for $i = 0, \ldots, m-1$,

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + \mu(t_i, \hat{X}_{t_i})[t_{i+1} - t_i] + \sigma(t_i, \hat{X}_{t_i})\sqrt{t_{i+1} - t_i} Z_{i+1}$$

with $Z_1, Z_2, \ldots Z_m$ being independent standard normal random vectors. [1] [BM01-a].

4.1.1 Simulation

In this section we will estimate the parameters of the Vasiček and Hull-White (Extended Vasiček) Models by using Euler approximation. After estimating the parameters of these models we will simulate the short rate.

Simulation of Vasiček Model

From the previous chapter remember that Vasiček Model under risk neutral probability Q for the short rate is;

$$dr_t = \beta(\alpha - r_t)dt + \sigma\, dW_t \tag{4.1.2}$$

and also we have found the analytic expression under risk neutral probability Q is as follows:

$$r_t = e^{-\beta(t-u)} r_u + \alpha(1 - e^{-\beta(t-u)}) + \sigma \int_u^t e^{-\beta(t-s)} dW_s. \tag{4.1.3}$$

Moreover, the conditional mean and the variance of r_t are

$$\forall\ u \leqslant t: \qquad E^Q\{r_t|\mathcal{F}_u\} = e^{-\beta(t-u)} r_u + \alpha(1 - e^{-\beta(t-u)}), \tag{4.1.4}$$

$$\begin{aligned} \forall\ u \leqslant t: \qquad Var\{r_t|\mathcal{F}_u\} &= E^Q[(\sigma \int_u^t e^{-\beta(t-s)} dW_s)^2|\mathcal{F}_u] & (4.1.5)\\ &= \frac{\sigma^2}{2\beta}[1 - e^{-2\beta(t-u)}]. & (4.1.6) \end{aligned}$$

Also as we said before, Vasiček assumed that the instantaneous spot rate under the real-world measure and risk-neutral measure as well evolves as an Ornstein-Uhlenbeck process with constant coefficients. Indeed, the observed prices are

[1] $N(0,1)$ is the standard normal distribution with mean zero and variance one.

under the real world which is represented by the probability P. Therefore, in order to do parameter estimation, we should consider the diffusion of Vasiček under P. If we just concern about the pricing of interest rate derivatives, we can directly model the short rate dynamics under the risk neutral measure Q, where λ will be implicit in our dynamics. The transition through probabilities can be done by using Girsanov Theorem. Note that in the remark at the preliminaries chapter, θ_t is interpreted as the market price of risk or risk premium while altering the probability measures. Our dynamic is in the risk neutral probability and we want to pass through the objective probability. If we add the market price of risk to the risk neutral values, we can obtain the equivalent values for objective probability. Let us assume θ_t is constant and equal to λ. Then,

$$
\begin{aligned}
L_t &= e^{\int_0^t \lambda dB_s - \frac{1}{2}\int_0^t \lambda^2 ds} \\
&= e^{\lambda B_t - \frac{1}{2}\lambda^2 t},
\end{aligned} \tag{4.1.7}
$$

where B_t is a standard Brownian motion under the probability P.
Claim: L_t is a martingale.

Proof. We calculate for all $s < t$:

$$
\begin{aligned}
E\left[e^{\lambda B_t - \frac{1}{2}\lambda^2 t} \middle| \mathcal{F}_s\right] &= E\left[e^{\lambda B_t - \frac{1}{2}\lambda^2 t}\, e^{\lambda B_s - \lambda B_s} \middle| \mathcal{F}_s\right] \\
&= e^{\lambda B_s - \frac{1}{2}\lambda^2 t}\, E[e^{\lambda(B_t - B_s)} | \mathcal{F}_s].
\end{aligned} \tag{4.1.8}
$$

Since Brownian motion has independent increments, (4.1.8) turns out to be

$$
\begin{aligned}
E\left[e^{\lambda B_t - \frac{1}{2}\lambda^2 t} \middle| \mathcal{F}_s\right] &= e^{\lambda B_s - \frac{1}{2}\lambda^2 t} E\left[e^{\lambda(B_t - B_s)}\right] \\
&= e^{\lambda B_s - \frac{1}{2}\lambda^2 t} E\left[e^{\lambda B_{t-s}}\right] \\
&= e^{\lambda B_s - \frac{1}{2}\lambda^2 t} e^{\frac{1}{2}\lambda^2 (t-s)} \\
&= e^{\lambda B_s - \frac{1}{2}\lambda^2 s} \qquad \square
\end{aligned} \tag{4.1.9}
$$

So, the sufficient condition for the Girsanov Theorem is satisfied. The standard

Brownian motion under risk neutral probability Q is just looking as follows:

$$dW_t = dB_t - \lambda\, dt.$$

Therefore, the Vasiček Model under the probability P is

$$dr_t = \beta(\alpha^* - r_t)dt + \sigma\, dB_t, \tag{4.1.10}$$

where $\alpha^* = \alpha - \frac{\lambda\sigma}{\beta}$.
Consequently, the conditional mean and the variance of r_t are

$$\forall\ u \leqslant t: \qquad E\{r_t|\,\mathcal{F}_u\} = e^{-\beta(t-u)}r_u + \alpha^*(1 - e^{-\beta(t-u)}), \tag{4.1.11}$$

$$\begin{aligned} \forall\ u \leqslant t: \qquad Var\{r_t|\,\mathcal{F}_u\} &= E\{(\sigma \int_u^t e^{-\beta(t-s)} dB_s)^2|\,\mathcal{F}_u\} && (4.1.12)\\ &= \frac{\sigma^2}{2\beta}\,[1 - e^{-2\beta(t-u)}]. && (4.1.13)\end{aligned}$$

The unconditional mean of the interest rate looks and behaves as follows:

$$E\,[r_t] = e^{-\beta t}r_0 + (1 - e^{-\beta t})\alpha^* \longrightarrow \alpha^* \quad \text{as } t \longrightarrow \infty \tag{4.1.14}$$

So, the process r_t has a limiting mean and we can consider α^* as *long run mean*. It has also limiting variance given by

$$Var[r_t] = \frac{\sigma^2}{2\beta}\,(1 - e^{-2\beta t}) \longrightarrow \frac{\sigma^2}{2\beta} \quad \text{as } t \longrightarrow \infty \tag{4.1.15}$$

Now, we are at the stage of discretization of the analytic solution of Vasiček

Model. To simulate r at times $t_1, \dots, t_n$, we define:

$$\begin{aligned} r_{t_{i+1}} &= e^{-\beta(t_{i+1}-t_i)} r_{t_i} + \alpha^*(1 - e^{-\beta(t_{i+1}-t_i)}) + \sigma\sqrt{\frac{1}{2\beta}(1 - e^{-2\beta(t_{i+1}-t_i)})}\, Z_{i+1} \\ &= e^{-\beta\Delta t} r_{t_i} + \alpha^*(1 - e^{-\beta(\Delta t)}) + \sigma\sqrt{\frac{1}{2\beta}(1 - e^{-2\beta(\Delta t)})}\, Z_{i+1} \\ r_{t_{i+1}} &= e^{-\beta} r_{t_i} + \alpha^*(1 - e^{-\beta}) + \sigma\sqrt{\frac{1}{2\beta}(1 - e^{-2\beta})}\, Z_{i+1}, \end{aligned} \tag{4.1.16}$$

where $\Delta t = t_{i+1} - t_i$ and since in simulation we use daily data we can take $\Delta t = 1$. Moreover, we can apply Euler approximation directly by using the discretization of stochastic differential equation of Vasiček Model which is slightly simpler Euler scheme. Then, we get

$$\begin{aligned} r_{t_{i+1}} - r_{t_i} &= (\beta\alpha^* - \beta r_{t_i})\Delta t + \sigma(\Delta t)^{\frac{1}{2}} Z_{i+1} \\ \rightarrow r_{t_{i+1}} &= \alpha^*\beta + (1-\beta) r_{t_i} + \sigma Z_{i+1}. \end{aligned} \tag{4.1.17}$$

Consider the following linear equation

$$X_t = x + aX_{t-1} + \xi, \tag{4.1.18}$$

where x is the constant term and ξ is just the innovation where has mean 0 and variance σ^2. Then the solution of (4.1.18) is

$$X_t = a^t \sum_{k=0}^{t} a^{-k}(\xi + x). \tag{4.1.19}$$

The unconditional mean of the solution of X_t is

$$\begin{aligned} E[X_t] &= a^t \sum_{k=0}^{t} a^{-k} E^Q[\xi + x] \\ &= a^t \sum_{k=0}^{t} a^{-k}\, E\,[x] \\ &= a^t \sum_{k=0}^{t} a^{-k}\, x \\ &= a^t\, x\, (1 + a^{-1} + a^{-2} + \ldots + a^{-t}) \\ &= x\, \frac{1 - a^{t+1}}{1 - a} \;\longrightarrow\; \frac{x}{1 - a}. \quad (t \to \infty) \end{aligned}$$

Furthermore, the unconditional variance is

$$\begin{aligned} Var[X_t] &= a^{2t} \sum_{k=0}^{t} a^{-2k} E[(\xi_k + x) - x]^2 \\ &= a^{2t} \sum_{k=0}^{t} a^{-2k} E\,[\xi_k^2] \\ &= a^{2t} \sum_{k=0}^{t} a^{-2k} \sigma^2 \\ &= \sigma^2 (1 + (a^2)^1 + (a^2)^2 + \ldots + (a^2)^t) \\ &= \sigma^2 \frac{1 - (a^2)^{t+1}}{1 - a^2} \;\longrightarrow\; \frac{\sigma^2}{1 - a^2} \quad (\text{as} t \to \infty). \end{aligned}$$

Since the discretization of Vasiček in the equation (4.1.17) is just the equation (4.1.18), the unconditional mean and variance of (4.1.17) look and behave as follows:

$$E[r_t] = (\alpha^*\, \beta)\, \frac{1 - (1 - \beta)^{t+1}}{\beta} \;\to\; \frac{\alpha^*\, \beta}{\beta} \quad \text{as t } \to \infty, \qquad (4.1.20)$$

$$Var[r_t] = \sigma^2 \frac{1 - (1 - \beta)^{2(t+1)}}{1 - (1 - \beta)^2} \;\to\; \frac{\sigma^2}{1 - (1 - \beta)^2} \quad \text{as t } \to\; \infty. \qquad (4.1.21)$$

In the parameter estimation, we use daily data and it is from 1 May 2001 to 16 March 2004. However, since the Turkey's financial market is not deep, there is lack of observed data. Therefore, we have to calculate zero coupon yields for the time to maturities that are not observed. In literature there are many methods which have been used to model the zero-coupon yield curve. We can put them into three categories: *spline based* models, *function based* models, and *stochastic* models. The most famous spline based models are McCulloch and FNZ (Fischer , Nychka and Zervos) models [BG02]. Popular function based models by practitioners are Nelson-Siegel [NS87], and Svensson [BS99] models. In fact, Svensson model is an extension of Nelson-Siegel model. In Nelson-Siegel model, a relatively simple function is postulated for the instantaneous forward curve. Svensson extended this work by altering the functional form of the instantaneous forward curve suggested by Nelson-Siegel. In this study, however, we concentrate on Nelson-Siegel model, which is less complex in the sense of number of parameters. Let us give some information about Nelson-Siegel methodology.

In their original work, Nelson and Siegel [BS99] proposed the instantaneous forward rate as a solution of a second order differential equations. Thus, they give the following solution for the instantaneous forward rate:

$$f(\theta) = \beta_0 + \beta_1 \, e^{-\frac{\theta}{\tau}} + \beta_2 \, \left[\frac{\theta}{\tau} \, e^{-\frac{\theta}{\tau}} \right]. \tag{4.1.22}$$

Since yield on the period $[t, T]$ is the average of sum of the rates which are active on the period $[t, t + \triangle t]$, by this simple intuition, yield $R(\theta)$ can be defined as the integral of (4.1.22). That is,

$$\begin{aligned} R(t,T) &= \frac{1}{T-t} \int_t^T f(s,T) ds \qquad (4.1.23) \\ &= \frac{1}{T-t} \int_t^T \left(\beta_0 + \beta_1 \, e^{-\frac{(T-s)}{\tau}} + \beta_2 \, \left[\frac{(T-s)}{\tau} \, e^{-\frac{(T-s)}{\tau}} \right] \right) ds \\ &= \beta_0 + (\beta_1 + \beta_2) \, \frac{[1 - e^{-\frac{T-t}{\tau}}]}{\frac{T-t}{\tau}} - \beta_2 \, e^{-\frac{T-t}{\tau}} \end{aligned}$$

$$
\begin{aligned}
&= \beta_0 + (\beta_1 + \beta_2)\,\frac{[1 - e^{-\frac{\theta}{\tau}}]}{\frac{\theta}{\tau}} - \beta_2\, e^{-\frac{\theta}{\tau}} \\
&= \frac{1}{\theta}\int_0^{\theta} \left(\beta_0 + \beta_1\, e^{-\frac{s}{\tau}} + \beta_2 \left[\frac{s}{\tau}\, e^{-\frac{s}{\tau}}\right]\right) ds \\
&= \frac{1}{\theta}\int_0^{\theta} f(s)ds \\
&= R(\theta) \qquad (4.1.24)
\end{aligned}
$$

where $\theta = T - t$ [BD04]. Now, let us analyze the structure of the yield and forward curves. The limiting value of $R(\theta)$, as θ approaches to infinity, is β_0, and as θ converges to 0 it is $(\beta_0 + \beta_1)$, which are necessarily the same as for the forward rate function since $R(\theta)$ is just an averaging of $f(\theta)$. Now, let us separate the forward rate function into three components: *long-term*, *medium-term* and lastly *short-term*. The long-term component is identified by the asymptotic value, β_0, of the function. The medium-term is identified by the functional component $\frac{\theta}{\tau}\, e^{-\frac{\theta}{\tau}}$ and for the designation of the short-term $e^{-\frac{\theta}{\tau}}$ is used.

In our model, the contributions of these three components are given by β_0 for long-term, β_1 for short-term and β_2 for medium-term. There are also some other features of the parameters that we should consider. If β_1 is negative, the forward curve will have a positive slope, and vice versa. Also, if β_2, as being the identifier of the magnitude and the direction of the hump, is positive, a hump will occur at τ whereas, if it is negative, a U-shaped value will occur at τ. Thus, we can conclude that τ is the parameter which is positive, specifies the position of the hump or U-shape on the entire curve. As a result, Nelson and Siegel have proposed that with appropriate choices of weights for these three components, it is possible to generate a variety of yield curves based on forward rate curves with monotonic and humped shapes [NS87]. To calibrate the Nelson-Siegel model, we are going to construct a sum of squared errors [HTF01]. Then, we are going to minimize this function with appropriate constraints and an initial

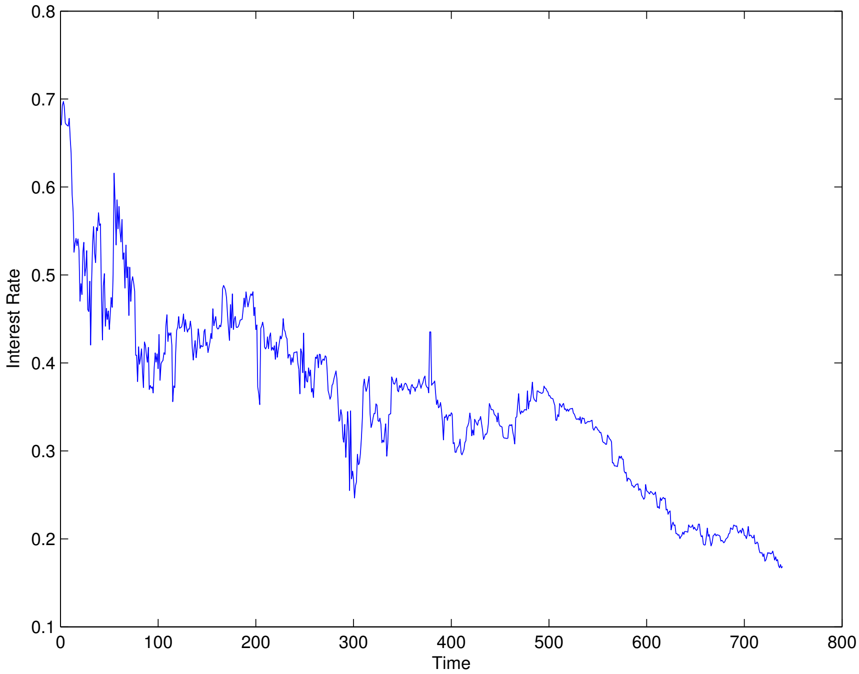

Figure 4.1: The plot of the real interest rate for the bonds having maturity 1 day

value. After the calibration, the performance of Nelson-Siegel yield curve will be measured by the value of the sum of squared errors. Finally, considering all these assumptions we obtain the time series of the zero coupon rates. However, the data we obtained consist in the nominal interest rate. To estimate the parameters we have to calculate the real interest rate. We can find the real interest rate by the following equation:

$$r_t = R_t - \pi^e_{t+1}, \tag{4.1.25}$$

where R is the nominal interest rate and π^e_{t+1} is the expected inflation rate for the next period. In this work, we assume that the inflation is anticipated; i.e, as the value of expected inflation rate we take just the value of the next period. The inflation rate data are obtained by the Turkish Central Bank.

We estimate the parameters from equation (4.1.16) and the equation (4.1.17) and then compare the estimated parameters. In both cases the estimation is based on ordinary *least square estimation.* In the first case we use regression function

$$r_{t_{i+1}} = \gamma_0 + \gamma_1 r_{t_i} + u_{t_{i+1}}; \tag{4.1.26}$$

where $\gamma_0 = \alpha^*(1 - e^{-\beta})$ is the intercept of the regression function,furthermore, $\gamma_1 = e^{-\beta}$ and the disturbance term $u_{t_{i+1}}$ is $\sigma\sqrt{\frac{1}{2\beta}(1 - e^{-2\beta})}Z_{i+1}$. In the second case, we use the regression function

$$r_{t_{i+1}} = \gamma_0 + \gamma_1 r_{t_i} + u_{t_{i+1}}, \tag{4.1.27}$$

where $\gamma_0 = \alpha^*\beta$ is the intercept of the regression function, $\gamma_1 = 1 - \beta$ and the disturbance term $u_{t_{i+1}}$ is σZ_{i+1}. The regressions of r_t on their first lag satisfy the usual assumptions. From the regression of (4.1.27) we obtain the estimation of parameters as follows:

$$\hat{\beta} = 0.0201$$
$$\hat{\alpha^*} = 0.3207$$

Similarly, from the regression of (4.1.26) we obtain the estimation of parameters as follows:

$$\hat{\beta} = 0.0203$$
$$\hat{\alpha^*} = 0.3207$$

There is only one parameter remained in Vasiček Model which is σ. We can estimate σ in the equation (4.1.16) by using asymptotic properties assuming that the number of observation is large enough. Remember that

$$Var[r_t] = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t}) \longrightarrow \frac{\sigma^2}{2\beta} \quad \text{as } t \longrightarrow \infty.$$

Then,

$$\sigma = \sqrt{Var[r_t] \times 2\beta}. \tag{4.1.28}$$

On the other hand, from the Euler discretization of (4.1.17) the estimation of σ is different by using the asymptotic properties. Remember that from this discretization we obtain

$$Var[r_t] = \frac{\sigma^2}{1-(1-\beta)^2}.$$

Then,

$$\sigma = \sqrt{Var[r_t] \times (1-\beta)^2}. \tag{4.1.29}$$

By using the equalities (4.1.28) and (4.1.29) we find the estimation of σ values 0.0203, 0.0201 respectively. As a summary, let us show the estimated values by a table:

By Euler Scheme	$\hat{\alpha^*}$	$\hat{\beta}$	$\hat{\sigma}$
From the equation (4.1.16)	0.3207	0.0203	0.0203
From the equation (4.1.17)	0.3207	0.0201	0.0201

Table 4.1: Estimated values by Euler scheme

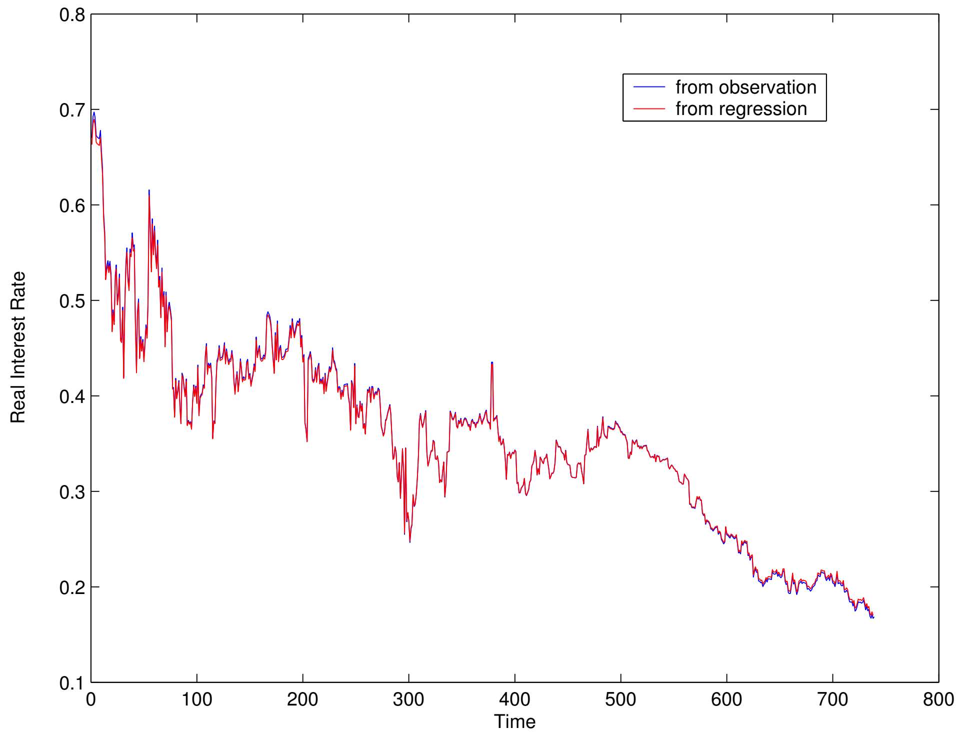

Figure 4.2: Comparison between the model and the actual data

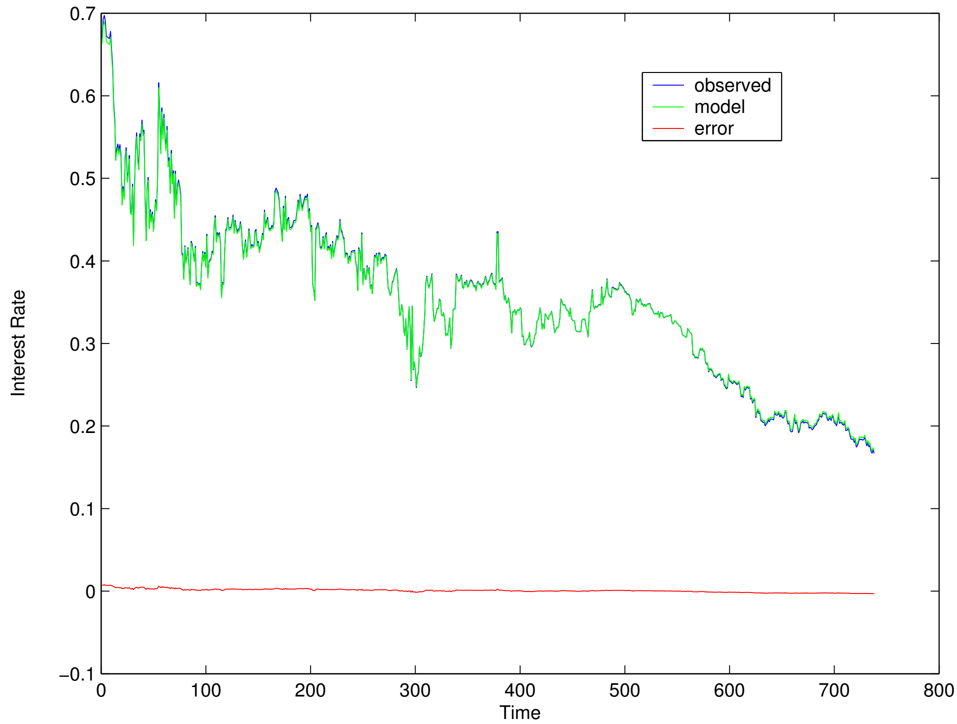

Figure 4.3: Error between the model and the actual data

Forecasting for the next day interest rate is done by Monte Carlo Method with 10000 simulations. From equation (4.1.16) and (4.1.17) we find the expected values of the zero coupon interest rate which have time to maturity one day for 17 March 2004.

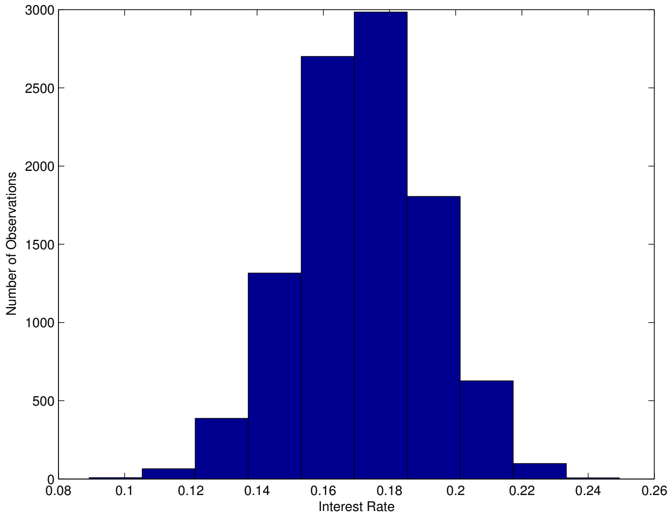

Figure 4.4: A histogram of predicted interest rate for 17 March 2004 by (4.1.16)

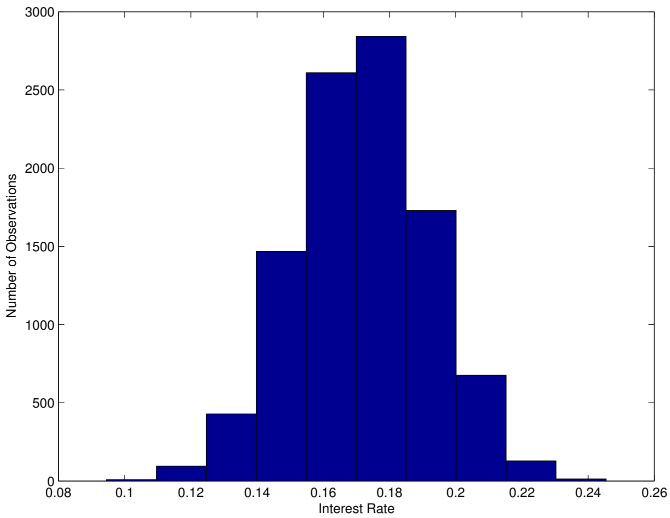

Figure 4.5: A histogram of predicted interest rate for 17 March 2004 by (4.1.17)

Also, from the equation (4.1.17) and by using Euler Scheme we find a probable one day interest rate for 17 March 2004 in the figure above. In our data the real interest rate for 17 March is indeed 15.38.

Simulation of Hull-White (Extended Vasiček) Model

In the previous section, we have already mentioned about the Hull-White model and its analytic expression. Remember that Hull and White (1990) extended the Vasiček Model by taking the parameters time varying. Moreover, we introduced $\theta(t)$ which is just for fitting the initial observed term structure. Remember that the diffusion term of the one factor short rate model of Hull-White under risk neutral probability Q is

$$dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dW_t.$$

From now on we assume that $\alpha(t)$ and $\sigma(t)$ are constants in order to simplify the model for practical purposes. So, the stochastic differential equation of Hull-White Model under risk neutral probability turns to

$$dr_t = (\theta(t) - \alpha r_t)dt + \sigma dW_t,$$

where the analytic expressions of r_t and $\theta(t)$ are

$$r_t = e^{-\alpha(t-u)}r_u + \int_u^t e^{-\alpha(t-s)}\theta(s)ds + \int_u^t e^{-\alpha(t-s)}\sigma dW_s, \tag{4.1.30}$$

and

$$\theta(t) = \frac{\partial f(0,t)}{\partial T} + \alpha f(0,t) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \tag{4.1.31}$$

Moreover, the conditional mean and the conditional variance of r_t are

$$\forall\, u \le t: \quad E^Q\left[r_t | \mathcal{F}_u\right] = r_u e^{-\alpha(t-u)} + E^Q\{\int_u^t \theta(s)e^{-\alpha(t-s)}ds|\mathcal{F}_u\} \tag{4.1.32}$$

$$\begin{aligned}\forall\, u \le t: \quad Var\{r_t | \mathcal{F}_u\} &= E^Q\left[(\sigma\int_u^t e^{-\alpha(t-s)}dW_s)^2|\mathcal{F}_u\right] \\ &= \frac{\sigma^2}{2\alpha}[1 - e^{-2\alpha(t-u)}].\end{aligned} \tag{4.1.33}$$

Now, we are faced with two problems. The first one is our dynamic is under risk neutral probability so we have to pass through to objective or real probability to estimate the deterministic parameters of the short rate. The second problem is the estimation of $\theta(t)$. The forward rate market has not developed in Turkey, so obviously we do not have forward rates for each day. To deal with this obstacle, we transform the zero coupon rates to forward rates.

Let us assume here market price of risk is constant and equal to λ. Then, by using Girsanov Theorem (Definition 2.10) we can say that the standard Brownian motion under objective probability measure is equal to

$$dB_t = dW_t + \lambda\, dt. \tag{4.1.34}$$

Therefore, the Hull-White (Extended Vasiček) Model under probability P is

$$dr_t = (\theta(t) - \alpha^* r_t)\, dt + \sigma\, dB_t, \tag{4.1.35}$$

where $\alpha^* = \alpha - \lambda$.
Then, the conditional mean and variance of r_t are

$$\forall\, u \leq t: \quad E^Q[r_t|\mathcal{F}_u] = r_u e^{-\alpha^*(t-u)} + E^Q[\int_u^t \theta(s) e^{-\alpha^*(t-s)} ds|\mathcal{F}_u] \tag{4.1.36}$$

$$\begin{aligned} \forall\, u \leq t: \quad Var\{r_t|\mathcal{F}_u\} &= E^Q[(\sigma \int_u^t e^{-\alpha^*(t-s)} dW_s)^2|\mathcal{F}_u] \\ &= \frac{\sigma^2}{2\alpha}[1 - e^{-2\alpha^*(t-u)}]. \end{aligned} \tag{4.1.37}$$

Moreover, θ under P probability measure is equal to as follows:

$$\theta(t) = \frac{\partial f(0,t)}{\partial T} + \alpha^* f(0,t) + \frac{\sigma^2}{2\alpha^*}(1 - e^{-2\alpha^* t}). \tag{4.1.38}$$

Simulating r_t by Euler approximation, we get:

$$r_{t_{i+1}} - r_{t_i} = (\theta(t_i) - \alpha^* r_{t_i})(t_{i+1} - t_i) + \sigma(t_{i+1} - t_i)^{1/2} Z_{i+1} \tag{4.1.39}$$

Since we use daily data we can take $t_{i+1} - t_i = \Delta t = 1$. Then the equation

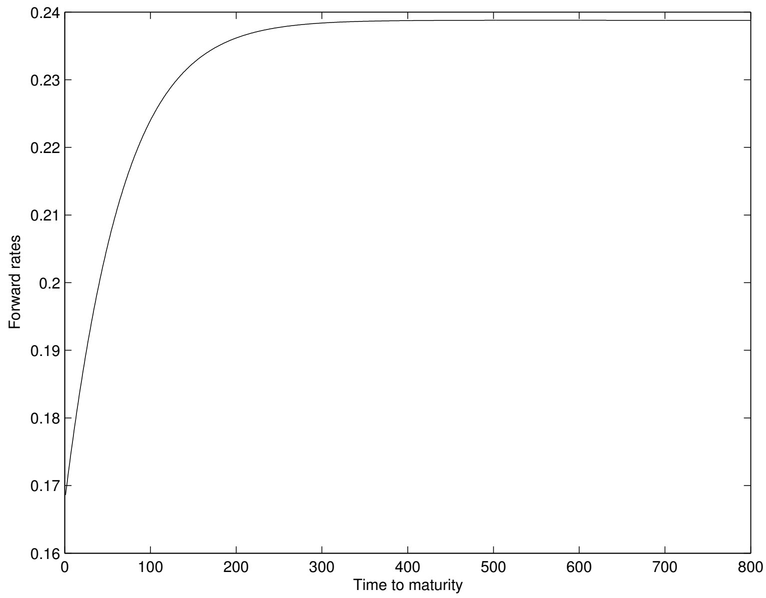

Figure 4.6: The forward rates obtained by zero coupon rates

(4.1.39) turns out to be:

$$r_{t_{i+1}} = \theta(t_i) + (1 - \alpha^*)r_{t_i} + \sigma Z_{i+1} \tag{4.1.40}$$

Also, we can disretize Hull-White (Extended Vasiček) model by the explicit solution of it. However, from table (4.1.1), we see that there is no big differences between the estimated values of parameters. Therefore, it is enough to apply Euler approximation directly to dynamic of Hull-White model as we did above.

The estimation of θ is done by using the equality (4.1.38). Note that, this equation implies that for each day there exists one θ and obtained by taking the average of the forward rates with respect to maturities. Moreover, while calculating θ, the α^* and σ terms are taken account. Hence, we must obtain the forward rates, α^* and σ. As we said before, in Turkey observed zero coupon rates for a day is limited. Therefore, we interpolate the zero coupon rates by using Nelson-Siegel (1987) methodology to obtain zero coupon rates for time to maturity 0 to 800. This methodology, generates curves that take on monotonic, humped or s-shapes depending on the parameters of the function Nelson-Siegel (1987) proposed. Then, we transform 16 March 2004 zero coupon rates to forward rates by using Matlab m-file **zero2fwd**. This function briefly works as follows: given a zero curve and a set of maturity dates as inputs, it returns an implied forward rate curve for the investment horizon represented by those maturity dates. Assume $\sigma = 0.0201$ and $\alpha^* = 0.0201$, then θ for 16 March 2004 is calculated. Moreover, by using equation (4.1.35), we predict 17 March 2004 zero coupon rate with time to maturity 1 day by Monte Carlo methodology.

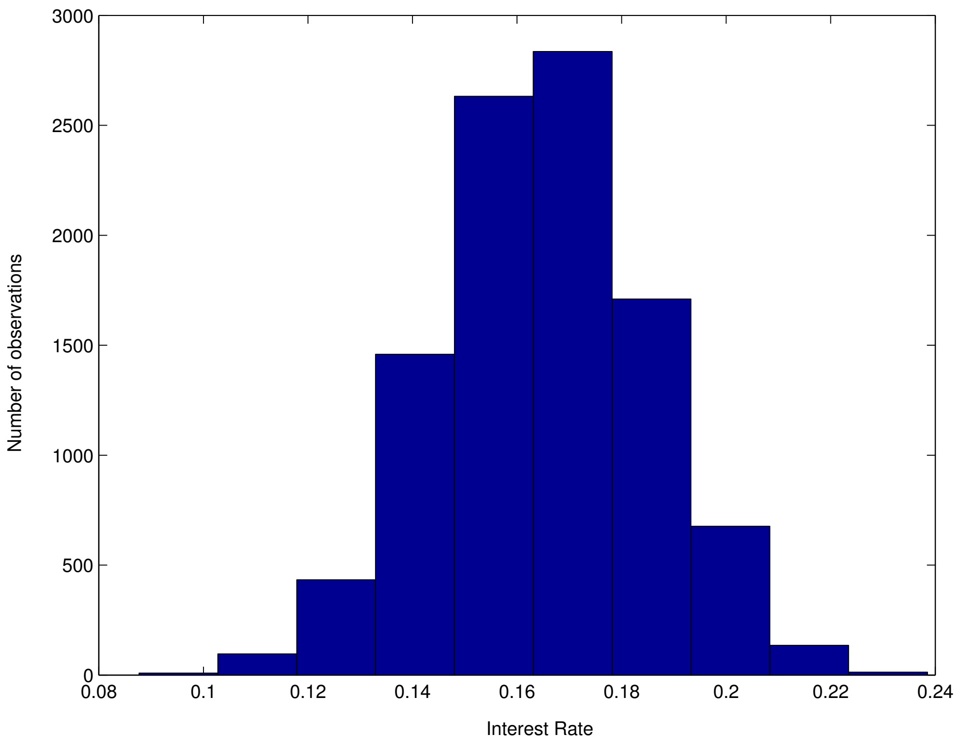

Figure 4.7: The predicted interest rate for 17 March 2004 by using Hull-White Model

4.2 Milshtein Scheme

First of all let's consider the process r_t satisfying the stochastic differential equation of the form

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t \tag{4.2.41}$$

with the initial value r_0 fixed. Define $\hat{r}$ which is the time discretized approximation to r. Approximation is on time grid $0 = t_0 < t_1 < \ldots < t_m$ where $\hat{r}_0 = r_0$ and for $i = 0, \ldots, m-1$,

$$\begin{aligned} \hat{r}_{t_{i+1}} &= \hat{r}_{t_i} + \mu(t_i, \hat{r}_{t_i})[t_{i+1} - t_i] + \sigma(t_i, \hat{r}_{t_i})\sqrt{t_{i+1} - t_i}Z_{i+1} \\ &+ \frac{1}{2}\sigma(t_i, \hat{r}_{t_i})\,\acute{\sigma}(t_i, \hat{r}_{t_i})(Z_{i+1}^2 - 1)[t_{i+1} - t_i], \end{aligned} \tag{4.2.42}$$

where $Z_1, Z_2, \ldots Z_m$ independent standard normal random vectors and $\acute{\sigma}$ is the derivative of σ w.r.t r_t. Milshtein scheme converges faster that Euler approximation. Also, note that the difference between two approximation is just the addition term:

$$\frac{1}{2}\sigma(t_i, \hat{r}_{t_i})\acute{\sigma}(t_i, \hat{r}_{t_i})(Z_{i+1}^2 - 1)[t_{i+1} - t_i]. \tag{4.2.43}$$

This method is applicable for the short rate models which has stochastic volatility. For instance, for the Vasiček Model $\sigma(t, r_t) \equiv \sigma$, so (4.2.43) is equal to zero and it turns to Euler approximation. Therefore, it is nonsense to use Milshtein approximation. Moreover, since (4.2.43) has expected value zero, it is usefulness to use Milshtein approximation for simulating interest rate process. For instance, for the CIR (1985), $\sigma(t, r_t) \equiv \sigma\sqrt{r_t}$, then (4.2.43) is equal to $\frac{1}{4}\sigma^2(Z_{t+1}^2 - 1)\Delta t \sim 10^{-5}\Delta t$ which is relatively small correction. Here, Δt is the time step between two discretized points.

Chapter 5

INTEREST RATE TREES

An interest rate tree is a discrete time representation of the stochastic process for the short rate in much the same way as a stock price tree is a discrete time representation of the process followed by a stock price. If the time step on the tree,i.e. $\triangle t$, then the rates on tree are continuously compounded $\triangle t$ period rates. The usual assumption when a tree is constructed is that the $\triangle t$ period rate, R, follows the same stochastic process as the instantaneous rate, r , in the corresponding continuous time model.

The main difference between interest rate trees and stock price trees is in the way how discounting is done. In a stock price tree, the discount rate is usually assumed to be the same at each node. In an interest rate tree, the discount rate varies from node to node. A trinomial tree is better than a binomial tree for interest rates. The main advantage of a trinomial tree is that it provides an extra degree of freedom, making it easier for the tree to represent features of the interest rate process such as mean reversion.

There are different types of branches in constructing trinomial interest rate tree:

- The first branching method is standard branching method. Here we think interest rate goes up, straight along or goes down.
- In the second branching method, the interest rate goes up by two, goes

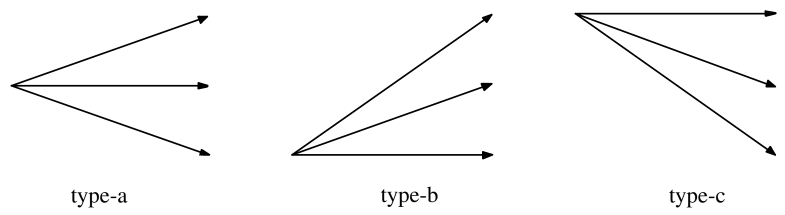

Figure 5.1: Types of branching in the trinomial tree

up by one and then straight along. When interest rates are very low, we use this branching method.

- In the third branching method, the interest rate goes down by two, goes down by one and then straight along. When interest rates are very high we use this branching method.

5.1 A General Tree-Building Procedure

Hull and White have proposed a robust two-stage procedure for constructing trinomial trees to represent a wide range of one-factor models [1]. We can apply this methodology to various kinds of instantaneous short rate models discussed in previous chapter. On the other hand, here we will use the Hull-White (Extended Vasicek) Model while constructing interest rate tree.

First Stage: As we know from the previous chapter, the Hull-White Model for the instantaneous short rate r is

$$dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dW_t,$$

where $\alpha(t), \theta(t), \sigma(t)$ are nonrandom functions of t [HW90].

[1]

- See Hull-White " Numerical Procedures for Implementing Term Structure Models I: Single Factor Models ", Journal of Derivatives , 2 ,(1994).
- See Hull-White "Using Hull-White Interest Rate Trees" , Journal of Derivatives, (1996).

The assumptions on tree buildings are the time step on the tree is constant which is equal to $\triangle t$ and $\triangle t$ period rate R, follows the same process as r, i.e.;

$$dR_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dW_t.$$

Clearly this is reasonable in the limit as $\triangle t \to 0$. To make calculations easier, we assume $\alpha(t)$ and $\sigma(t)$ are constants, say, α and σ.
The first stage in building a tree for this model is to construct a tree for a variable R^* that is initially zero and follows the process:

$$dR^* = -\alpha R^* dt + \sigma dW_t.$$

This process is symmetrical about $R^* = 0$. The variable $R^*_{t+\triangle t} - R^*_t$ is normally distributed. The mean and variance of this process are as follows:

$$\begin{aligned} E[R^*_{t+\triangle t} - R^*_t] &= -\alpha R^*_t \triangle t, \\ Var[R^*_{t+\triangle t} - R^*_t] &= \sigma^2 \triangle t. \end{aligned}$$

We define $\triangle R^*$ as spacing between interest rates on the tree and set

$$\triangle R^* := \sigma\sqrt{3\triangle t}. \tag{5.1.1}$$

This proves to be a good choice of $\triangle R^*$ from the viewpoint of error minimization. To build a tree, firstly we must resolve which of the tree branching methods (type a, type b or type c) will apply at each node. This will determine the *overall geometry* of the tree. Once this is done, the branching probabilities need to be calculated. Define (i, j) as the node where $t = i\triangle t$ and $R^* = j\triangle R^*$. Also note that i should be positive integer whereas j is positive or negative integer. In a tree-building procedure for R^*, firstly we must identify the altitude of the tree by defining j_{max} and j_{min}. Define j_{max} as the value of j where we switch from branching a to branching c and j_{min} as the value of j where we switch from branching a to b. Hull and White show that the probabilities are

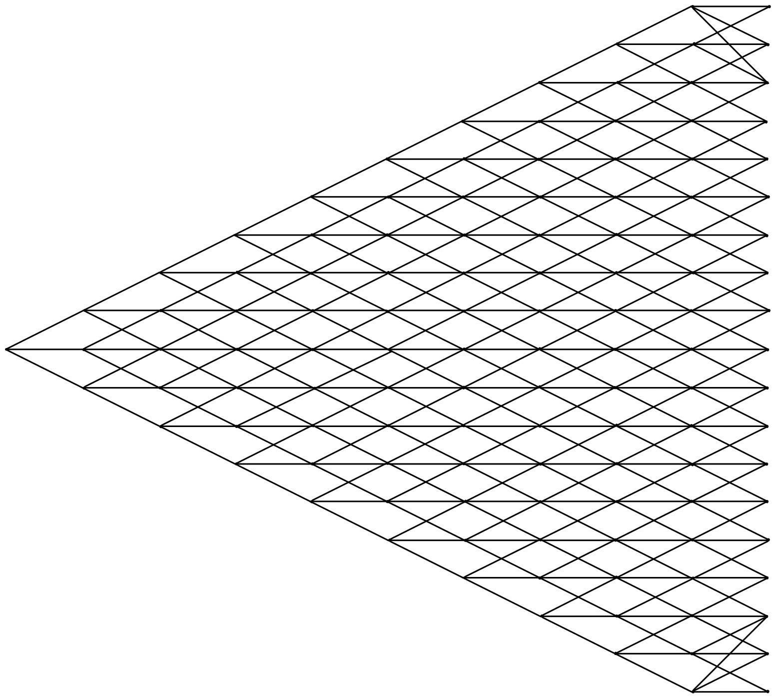

Figure 5.2: Overall geometry of a tree

always positive if we set j_{max} equal to the smallest integer greater than $\frac{0.184}{\alpha\triangle t}$ and $j_{min} = -j_{max}$[2]. Define probabilities p_u, p_m, p_d as the probabilities of the highest, middle and lowest branches emanating from the node. Here, we use *methods of moments approach*,i.e. the probabilities are chosen to match *expected change* and *variance of the change* in R^* over the next time interval $\triangle t$. The probabilities must also sum to unity. This leads to three equations for each type of branches; for (i, j) node we can write the equations for identifying probabilities;

For branching type-a:

$$\begin{aligned} p_u\triangle R^* - p_d\triangle R^* &= -\alpha j\triangle R^*\triangle t, \\ p_u\triangle R^{*2} + p_d\triangle R^{*2} &= \sigma^2\triangle t + \alpha^2 j^2\triangle R^{*2}\triangle t^2, \\ p_u + p_m + p_d &= 1. \end{aligned}$$

Using equation (5.1.1), we get the following probabilities which are just func-

[2]Hull-White also argue that j_{max} is an integer between $\frac{0.184}{\alpha\triangle t}$ and $\frac{0.816}{\alpha\triangle t}$.

tions of α, j and $\triangle t$:

$$\begin{aligned}
p_u &= \frac{1}{6} + \frac{\alpha^2 j^2 \triangle t^2 - \alpha j \triangle t}{2}, \\
p_m &= \frac{2}{3} - \alpha^2 j^2 \triangle t^2, \\
p_d &= \frac{1}{6} + \frac{\alpha^2 j^2 \triangle t^2 + \alpha j \triangle t}{2}.
\end{aligned}$$

<u>For branching type-b:</u>

$$\begin{aligned}
2p_u \triangle R^* + p_m \triangle R^* &= -\alpha j \triangle R^* \triangle t, \\
4p_u \triangle R^{*2} + p_m \triangle R^{*2} &= \sigma^2 \triangle t + \alpha^2 j^2 \triangle R^{*2} \triangle t^2, \\
p_u + p_m + p_d &= 1.
\end{aligned}$$

Again using $\triangle {R^*}^2 = 3\sigma^2 \triangle t$, we get the following probabilities:

$$\begin{aligned}
p_u &= \frac{1}{6} + \frac{\alpha^2 j^2 \triangle t^2 + \alpha j \triangle t}{2}, \\
p_m &= -\frac{1}{3} - \alpha^2 j^2 \triangle t^2 - 2\alpha j \triangle t, \\
p_d &= \frac{7}{6} + \frac{\alpha^2 j^2 \triangle t^2 + 3\alpha j \triangle t}{2}.
\end{aligned}$$

<u>For branching type-c:</u>

$$\begin{aligned}
-p_m \triangle R^* - 2p_d \triangle R^* &= -\alpha j \triangle R^* \triangle t, \\
p_m \triangle R^{*2} + 4p_d \triangle R^{*2} &= \sigma^2 \triangle t + \alpha^2 j^2 \triangle R^{*2} \triangle t^2, \\
p_u + p_m + p_d &= 1.
\end{aligned}$$

Then we have the following probabilities for this type of branching:

$$\begin{aligned}
p_u &= \frac{7}{6} + \frac{\alpha^2 j^2 \triangle t^2 - 3\alpha j \triangle t}{2}, \\
p_m &= -\frac{1}{3} - \alpha^2 j^2 \triangle t^2 + 2\alpha j \triangle t, \\
p_d &= \frac{7}{6} + \frac{\alpha^2 j^2 \triangle t^2 - \alpha j \triangle t}{2}.
\end{aligned}$$

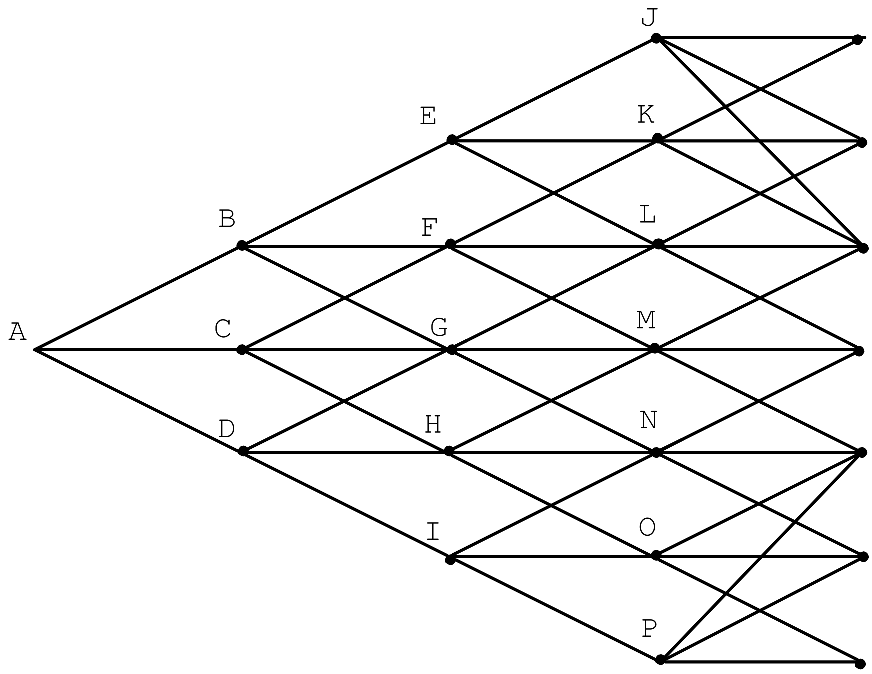

Figure 5.3: Tree for R^* in Hull-White Model

Note that the probabilities at each node depend only on j. So if two nodes on the same horizontal line and the branches are the same, then we can conclude that the probabilities are the same, which means that probabilities are time homogeneous on the tree. Furthermore, tree is symmetrical and the probabilities of a node with its symmetry node with respect to the center is just the mirror image of the other's probabilities. To understand better, let us illustrate the first stage. In order to define the values of $j's$ we have to determine the values of $\triangle t$ and α. Assume $\triangle t$ period is 1 year and it is constant. Also, in order to determine the $\triangle R^*$, we have to identify σ. The parameters α and σ need to be determined by the dynamics of instantaneous short rate by using observed data. Assume that $\alpha = 0.08$, $\sigma = 0.01$. Let us start with calculating $\triangle R^*$ by using the equation (5.1.1):

$$\triangle R^* = \sigma\sqrt{3\triangle t} = 0.01\sqrt{3} = 0.0173. \tag{5.1.2}$$

Since j_{max} is the smallest integer greater than $\frac{0.184}{\alpha\triangle t} = \frac{0.184}{0.08}$, then j_{max} is equal to 3 and, immediately, j_{min} is equal to -3. After determining the j values we can construct overall geometry of the tree with starting $R^* = 0$. Since the probabilities at each node depend only on j we can easily conclude that the probabilities at node E and K are same. Similarly since B, F and L have the same j values and branching types, their probabilities are identical. The nodes A, C, G and M have same probabilities, too. Moreover, since our tree is symmetric, the probabilities at the nodes B and D are just mirror images. Furthermore, E and I, J and P, etc., have the same probability law as shown in the below table.

Node	A	B	C	D	E	F	G	H
R^*	0.000%	1.732%	0.000%	-1.732%	3.464%	1.732%	0.000%	-1.732%
p_u	0.1667	0.1299	0.1667	0.2099	0.0995	0.1299	0.1667	0.2099
p_m	0.6667	0.6603	0.6667	0.6603	0.6411	0.6603	0.6667	0.6603
p_d	0.1667	0.2099	0.1667	0.1299	0.2595	0.2099	0.1667	0.1299
Node	**I**	**J**	**K**	**L**	**M**	**N**	**O**	**P**
R^*	-3.464%	5.196%	3.464%	1.732%	0.000%	-1.732%	-3.464%	-5.196
p_u	0.2595	0.8355	0.0995	0.1299	0.1667	0.2099	0.2595	0.0755
p_m	0.6411	0.0891	0.6411	0.6603	0.6667	0.6603	0.6411	0.0891
p_d	0.0995	0.0755	0.2595	0.2099	0.1667	0.1299	0.0995	0.8355

Table 5.1: R^* values and the probabilities at each node

Second Stage: The second and the last stage of the tree building procedure is converting the tree R^* to the tree R. The randomness comes from this stage. We change the overall geometry of the tree by defining a function which is the difference between R and R^*, i.e.,

$$\beta_t = R_t - R_t^*. \tag{5.1.3}$$

Since R follows the Hull-White Model and R^* satisfies Ornstein-Uhlenbeck process, it follows that

$$d\beta_t = [\theta(t) - \alpha\beta_t]\, dt. \tag{5.1.4}$$

In Chapter 3, we calculate the value of $\theta(t)$ which is equal to equation (4.1.31). Therefore, β_t is calculated from (5.1.4) by solving the ordinary differential equation and substituting the value of $\theta(t)$, then the result is

$$\beta_t = f(0,t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2. \tag{5.1.5}$$

On the other hand, calculating β for each time step is not an appropriate way for converting the tree. Indeed, we can calculate β iteratively by using the zero coupon bond rates. This approach is more useful because β can be calculated in the situations that initial zero coupon bonds are not differentiable. To understand better, let us continue with our numeric example shown in the first stage. Let us for each time step define $\beta_{i\,\Delta t}$ which is just the value of R-tree minus the value of R^*-tree at time $i\,\Delta t$. Also, define $Q_{i,j}$ as the present value of the zero coupon bond if node (i,j) is reached, and 0 otherwise. Therefore, the values β and Q's are calculated using forward induction in such a way that the initial term structure is fitted exactly. Note that $Q_{0,0}$ is equal to 1 (since the value of the zero coupon bond is just equal to its face value) and set β_0 as the initial Δperiod rate. Let us assume that the zero coupon bond rates for one, two, three and four years are 3.25%, 4.15 %, 4.75 % and 5.4 %, respectively. Since we calculate the probability of reaching the nodes, then we can easily calculate $Q_{1,1}, Q_{1,0}, Q_{1,-1}$ which are the present value of the zero coupon bonds having time to maturity one year at nodes B, C, D respectively. Then, $Q_{1,-1} = Q_{1,1} = 0.1667e^{-0.0325} = 0.1614 and Q_{1,0} = 0.6666e^{-0.0325} = 0.6453$. Since R is equal to at each time step $\beta + \Delta R$, then β_1 can be calculated by the following equation:

$$Q_{1,1}e^{-(\beta_1+0.01732)} + Q_{1,0}e^{-\beta_1} + Q_{1,-1}e^{-(\beta_1-0.01732)} = 0.9204, \tag{5.1.6}$$

where $0.9204 = e^{-0.0415\times 2}$. Now, we can find $\beta_1 = 0.0505$ by solving the equation (5.1.6). Then we have to calculate Q values to find β_2. Consider $Q_{2,2}$ which is the value of the zero coupon bond if node F is reached, and 0 otherwise. Node F is reached only from nodes B and C. Therefore, first of all we have to calculate the value of the security at each node then discount to present by using the value of $Q_{1,1}$ and $Q_{1,-1}$. Therefore, the calculations of Q's are shortened by using the previous Q values. Using iteration we can calculate $Q_{2,2}, Q_{2,1}, Q_{2,0}, Q_{2,-1}, Q_{2,-2}$ by the following equations:

$$\begin{aligned}
Q_{2,2} &= p_u^B e^{-r_B} Q_{1,1}, \\
Q_{2,1} &= p_m^B e^{-r_B} Q_{1,1} + p_u^C e^{-r_C} Q_{1,0}, \\
Q_{2,0} &= p_d^B e^{-r_B} Q_{1,1} + p_m^C e^{-r_C} Q_{1,0} + p_u^D e^{-r_D} Q_{1,-1}, \\
Q_{2,-1} &= p_d^C e^{-r_C} Q_{1,0} + p_m^D e^{-r_D} Q_{1,-1}, \\
Q_{2,-2} &= p_d^D e^{-r_D} Q_{1,-1}.
\end{aligned}$$

Indeed, we can represent the equalities into matrix form:

$$\begin{pmatrix} Q_{2,2} \\ Q_{2,1} \\ Q_{2,0} \\ Q_{2,-1} \\ Q_{2,-2} \end{pmatrix} = \begin{pmatrix} p_u^B & 0 & 0 \\ p_m^B & p_u^C & 0 \\ p_d^B & p_m^C & p_u^D \\ 0 & p_d^C & p_m^D \\ 0 & 0 & p_d^D \end{pmatrix} \begin{pmatrix} e^{-r_B} Q_{1,1} \\ e^{-r_C} Q_{1,0} \\ e^{-r_D} Q_{1,-1} \end{pmatrix}$$

where $r_B = \beta_1 + 0.01732 = 0.06782$, $r_C = \beta_1 = 0.0505$ and $r_D = \beta_1 - 0.01732 = 0.03318$. So we have found that the Q values are: $Q_{2,2} = 0.0196, Q_{2,1} = 0.2019$, $Q_{2,0} = 0.4734, Q_{2,-1} = 0.2054, Q_{2,-2} = 0.0203$. Then, β_2 is just the value that satisfies the equality below:

$$\begin{aligned}
\beta_2 &= \ln[Q_{2,2}e^{-2\Delta R} + Q_{2,1}e^{-\Delta R} + Q_{2,0} + Q_{2,-1}e^{\Delta R} + Q_{2,-2}e^{2\Delta R}] - \ln P_3 \\
&= 0.06, \qquad (5.1.7)
\end{aligned}$$

where P_3 is the value of the zero coupon bond which has time to maturity 3 years. So we can calculate it by discounting 1 dollar with the 3 year zero coupon rate, 4.75 %.

By using the same methodology, we calculate the Q values for the next nodes which satisfies the following equality;

$$\begin{pmatrix} Q_{3,3} \\ Q_{3,2} \\ Q_{3,1} \\ Q_{3,0} \\ Q_{3,-1} \\ Q_{3,-2} \\ Q_{3,-3} \end{pmatrix} = \begin{pmatrix} p_u^E & 0 & 0 & 0 & 0 \\ p_m^E & p_u^F & 0 & 0 & 0 \\ p_d^E & p_m^F & p_u^G & 0 & 0 \\ 0 & p_d^F & p_m^G & p_u^H & 0 \\ 0 & 0 & p_d^G & p_m^H & p_u^I \\ 0 & 0 & 0 & p_d^H & p_m^I \\ 0 & 0 & 0 & 0 & p_d^I \end{pmatrix} \begin{pmatrix} e^{-r_E}\, Q_{2,2} \\ e^{-r_F}\, Q_{2,1} \\ e^{-r_G}\, Q_{2,0} \\ e^{-r_H}\, Q_{2,-1} \\ e^{-r_I}\, Q_{2,-2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} Q_{3,3} \\ Q_{3,2} \\ Q_{3,1} \\ Q_{3,0} \\ Q_{3,-1} \\ Q_{3,-2} \\ Q_{3,-3} \end{pmatrix} = \begin{pmatrix} 0.0018 \\ 0.0358 \\ 0.2028 \\ 0.3779 \\ 0.2094 \\ 0.0382 \\ 0.0020 \end{pmatrix}$$

After that we can easily calculate β_3 by similar method mentioned above. It is equal to 0.0787. Finally, we obtain the R values which concludes our example. Therefore, by this methodology we approximate the Hull-White model and represent the zero coupon rates in three years where time step is 1 year.

Node	A	B	C	D	E	F	G	H
R	3.250%	6.782%	5.050%	3.318%	9.464%	7.732%	6.000%	4.268%
p_u	0.1667	0.1299	0.1667	0.2099	0.0995	0.1299	0.1667	0.2099
p_m	0.6667	0.6603	0.6667	0.6603	0.6411	0.6603	0.6667	0.6603
p_d	0.1667	0.2099	0.1667	0.1299	0.2595	0.2099	0.1667	0.1299

Node	I	J	K	L	M	N	O	P
R	2.536%	13.067%	11.33%	9.602%	7.870%	6.138%	4.406%	2.674
p_u	0.2595	0.8355	0.0995	0.1299	0.1667	0.2099	0.2595	0.0755
p_m	0.6411	0.0891	0.6411	0.6603	0.6667	0.6603	0.6411	0.0891
p_d	0.0995	0.0755	0.2595	0.2099	0.1667	0.1299	0.0995	0.8355

Table 5.2: R values and the probabilities at each node

The other method for calculating β's is using the equation (5.1.5). The biggest problem in that situation is obtaining *forward rates*. For example, in Turkey we do not have forward rates for zero coupon bonds. In this example, we transform the given zero coupon rates to forward rates by the definition of the rates mentioned in the preliminaries part and the software Matlab. After acquiring the forward rates, it is easy to find the β values by equation (5.1.5):

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0.0325 \\ 0.0506 \\ 0.0596 \\ 0.0737 \end{pmatrix}$$

We shift the R^* values by this β values and obtain some results. To compare better, let us show the table of values of R at each nodes obtained from two methodology:

	R^1	R^2
A	3.250%	3.250%
B	6.782%	6.792 %
C	5.050%	5.060 %
D	3.318%	3.328%
E	9.464%	9.424%
F	7.732%	7.692%
G	6.00%	5.96%
H	4.268%	4.228%
I	2.536%	2.496%
J	13.066%	12.566%
K	11.334%	10.834%
L	9.602%	9.102%
M	7.870%	7.370%
N	6.138%	5.638%
O	4.406%	3.906%
P	2.674 %	2.174%

Table 5.3: R values using different β estimation methods

Here, R^1 represents the value of R's obtained by the the first method where β's are found iteratively and R^2 is obtained from R^* shifted by the β's obtained directly by the equation (5.1.5).

5.2 Implementation

In this section, we will implement the trinomial interest rate tree methodology for zero coupon rates in Turkey by using Hull-White Model. Since we have mentioned a lot about the method we briefly give the results. In this application, we take the time step 1 month since we consider $\triangle t$ in terms of years, so $\triangle t = \frac{1}{12}$. On the other hand, the zero coupon rates are annualized which means that they are assumed to be annually compounding. Therefore, we have to convert them to equivalent rates compounded monthly since our time horizon is monthly. Also, since in Turkey forward market has not developed yet we do not have forward rates for bonds. Therefore, we have to transform to zero coupon rates to forward rates. We convert the data by using Matlab m-file **zero2fwd**. Suppose we are on 16 March 2004 and we want to model the zero coupon rates for the next months. Also, assume

$$\alpha = 0.0203$$

$$\sigma = 0.0203$$

then,

$$\Delta R^* = \sigma\sqrt{3\triangle t} = 0.0203 \times \sqrt{\frac{1}{4}} = 0.0144$$

For constituting the overall geometry of the trinomial tree, we have to determine the j values which form the level of the tree. Note that Hull and White (1990) proposed j_{max} as the smallest integer greater than $\frac{0.184}{\alpha\triangle t}$ so:

$$j_{max} = 109$$

which means that by these values we can model the zero coupon rates for the next 109 months. But here we will build up the tree for 3 months only. Since

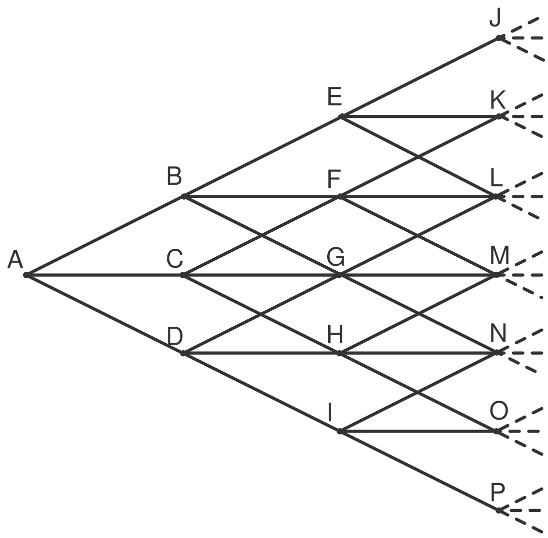

Figure 5.4: Trinomial tree of R^* with $\triangle t = 1/12$

we assume that the probabilities at each j values are equal which implies *stationarity*. There is no need to tabulate the probabilities for each node. Instead we give the probabilities for each level:

j values	$j=3$	$j=2$	$j=1$	$j=0$	$j=-1$	$j=-2$	$j=-3$
p_u	0.1641	0.1650	0.1658	0.1667	0.1675	0.1684	0.1692
p_m	0.6666	0.6667	0.6667	0.6667	0.6667	0.6667	0.6666
p_d	0.1692	0.1684	0.1675	0.1667	0.1658	0.1650	0.1641

Table 5.4: Probabilities at each level

Now we are at the stage of converting the R^* tree to R tree which is done by the β values for each time. In the previous section we have given two methods for obtaining β. For practical purposes we obtain β directly rather than obtaining iteratively. By using the equation (5.1.5) we get

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0.0678 \\ 0.0714 \\ 0.0780 \\ 0.0860 \end{pmatrix}$$

Then, by using calculated β values we can alter R^* tree to R tree easily.

	R^*	R
A	0.00%	6.78%
B	1.44%	8.58 %
C	0.000%	7.14 %
D	-1.44%	5.70%
E	2.88%	10.68%
F	1.44%	9.24%
G	0.00%	7.80%
H	-1.44%	6.36%
I	-2.88%	4.92%
J	4.32%	12.92%
K	2.88%	11.48%
L	1.44%	10.04%
M	0.00%	8.60%
N	-1.44%	7.16%
O	-2.88%	5.72%
P	-4.32 %	4.28%

Table 5.5: The calculated R^* and R values

where R values are the real zero coupon interest rates that are monthly compounded.

Chapter 6

CONCLUSION

In this work we reviewed some theoretical and practical aspects of the Interest Rate Theory. We gave a review of the one-factor short rate models that are commonly encountered in literature. As far as we know, there is no book containing a sufficiently complete list of existing one-factor short rate models and their analytic solutions. We have thus tried to complete this gap. In addition, after having considered the discretization techniques consisting of the Euler and Milshtein approximations, we adapted some of these models to Turkey's financial market data and estimated the parameters of the models. We finally adapted the trinomial interest rate tree technique to the Hull-White (Extended Vasicek) model to represent the evolution of Turkey's zero coupon rates.

Our future research will be on further developments of the short rate models and some modifications on the models that fit Turkey's financial market data. Furthermore, modeling and implementing one-factor short rate models with Gaussian random field instead of a single Brownian motion is also a possible direction for further studies.

Option pricing is another topic that we may deal with in further studies. There are closed form expressions of option prices in terms of the existing one-factor short rate models. It is also possible to use trinomial interest rate trees for option pricing. In this field the future research will depend on the development of the option pricing needs for Turkey's financial markets.

References

[BD04] Bayazıt D., *Yield Curve Estimation and Prediction with Vasiček Model*, Master Thesis, (2004).

[BG02] Bolder D. and Gusba S., *Exponentials, polynomials, and fourier series: more yield curve modelling at the Bank of Canada*, Working Paper, (2002)

[BS99] Bolder D. and Streliski D., *Yield curve modelling at the Bank of Canada*, Technical Report, 84, (1999).

[BDT90] Black F., Derman E. and Toy W., *A one-factor model of interest rates and its application to treassury bond options*, Financial Analysts Journal, January-February (1990), 33-39.

[BS73] Black F. and Scholes M., *The pricing of options and corporate liabilities*, Journal of Political Economy, 81 (1973), 637-654.

[BK91] Black F. and Karasinski P., *Bond option pricing when short rates are lognormal*, Financial Analysts Journal, July-August (1991), 52-59.

[Bjo98] Björk T., *Arbitrage Theory in Continuous Time*, Oxford University Press (1998).

[BM01-a] Brigo D. and Mercurio F., *Interest Rate Models Theory and Practice*, Springer (2001).

[BM01-b] Brigo D. and Mercurio F., *A Deterministic-shift extension of analytically-tractable and time-homogeneous short-rate models*, Finance and Stochastic, 5 (2001), 369-387.

[CKLS] Chan C. K., Karolyi A. G., Longstaff A. F. and Sanders B. A., *The Volatility of Short-Term Interest Rates: An Empirical Comparasion of Alternative Models of the Term Structure of Interest Rates*, The Ohio State University, 1998.

[Cha98] Chance M.D., *An Introduction to Derivatives*, Fourth Edition, Dryden Press (1998).

[CYY99] Chu K.L., Yang H. and Yuen K.C., *Estimation in the constant elasticity of variance Model*, preprint, Department of Statistics and Actuarial Science, University of Hong Kong, 1999.

[Cou82] Courtadon G. , *The Pricing of Options on Default-Free Bonds*,Journal of Financial and Quantative Analysis,17 (1982),75-100.

[Cox75] Cox J., *Notes on option pricing I: constant elasticity of diffusions*, draft of a paper, Stanford University, (1975).

[CR76] Cox J. and Ross S., *The valuation of options for alternative stochastic processes*, Journal of Financial Economics, 3 (1976), 145-166.

[CIR85] Cox C.C., Ingersoll E.J. and Ross A.S., *A theory of the term structure of interest rates*, Econometrica, 53 (1985), 385-407.

[DJ03] Dana R.A. and Jeanblanc M., *Financial Markets in Continuous Time*, Springer (2003).

[Dot78] Dothan L.U., *On the Term Structure of Interest Rates*, Journal of Financial Economics, (1978), 59-69.

[Glas04] Glasserman P., *Monte Carlo Methods in Financial Engineering*, Springer (2004).

[HTF01] Hastie R., Tibshirami R. and Friedman J., *The Elements of Statistical Learning*, Springer(2001).

[HL86] Ho S.Y.T. and Lee S. , *Term structure movements and pricing interest rate contingent claims*, Journal of Finance,41 (1986), 1011-1029.

[HW90] Hull J. and White A., *Pricing Interest Rate Derivative Securities*, The Review of Financial Studies, 3 (1990), 573-592.

[HW93] Hull J. and White A., *One-factor interest rate models and the valuation of interest-rate derivative securities*, Journal of Financial and Quantative Anaysis, 28 (1993), 235-254.

[JW02] James J. and Webber N., *Interest Rate Modelling*, Wiley (2002).

[Kle01] Klebaner C.F., *Introduction to Stochastic Calculus with Applications*, Chapman&Hall Press (2000).

[KP92] Kloeden E.P. and Platen E., *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag (1992).

[KH01] Körezlioğlu H. and Hayfavi A., *Elements of Probability Theory*, METU (2001).

[Kwo98] Kwok K.Y., *Mathematical Models of Financial Derivatives*, Springer (1998).

[LL00] Lamberton D. and Lapeyre B., *Introduction to Stochastic Calculus*, CRC Press (2000).

[MR83] Marsh A. T and Rosenfeld R. E., *Stochastic processes for interest rates and equilibrium bond prices*, Journal of Finance, 38 (1983), 635-646.

[MM00] Mercurio F. and Moraleda M. J., *An analytically tractable interest rate model with humped volatility*, European Journal of Operational Research , 120 (2000), 205-214.

[NS87] Nelson C.R. and Siegel A.F., *Parsimonious modelling of yield curves.*, Journal of Business, 60 (1987), 473-489.

[Plis97] Pliska R.S., *Introduction to Mathematical Finance*, Blackwell (1997).

[RM80] Rendleman R. and Bartter B., *The pricing of options on debt securities*, Journal of Financial and Quantative Analysis, 15 (1980), 11-24.

[Sch03] Schoutens W., *Lévy Processes in Finance*,Wiley (2003).

[Sey02] Seydel R., *Tools for Computational Finance*, Springer (2002).

[TG03] Treepongkaruna S. and Gray S., *Short-term interest rate models: valuing interest rate derivatives using a Monte-Carlo approach*, Accounting and Finance, 43 (2003), 231-259.

[Vas77] Vasicek O., *An equilibrium characterization of the term structure*, Journal of Financial Econometrics, 5 (1977), 177-188.

[Yao00] Yao Y., *Term Structure Models: A Perspective from the Long Rate*, CRC Press(2000).