The Interval Shapley Value: An Axiomatization

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Abstract

In this paper we study properties of the interval Shapley value on the class of size monotonic interval games, and axiomatically characterize its restriction to a special subclass of cooperative interval games by using the properties of additivity, efficiency, symmetry and dummy player.

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1 Introduction

The Shapley value (Shapley (1953)) is one of the most interesting solution concepts in cooperative game theory. Inspired by some equivalent formulations of it, we focus on the restriction of the interval Shapley value to a subclass of cooperative interval games and give an axiomatization on this subclass. Interval games in the subclass we consider arise from several economic and Operations Research situations with interval data. For example

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peer group games (Branzei, Mallozzi and Tijs (2008)) are useful in modeling sequencial production situations, auctions and flow situations. A similar subclass of interval games arises from airport situations with interval data (Alparslan Gök, Branzei and Tijs (2009)), and the interval Shapley value of an airport interval game coincides with the interval Baker-Thompson allocation.

The paper is organized as follows. In Section 2 we recall basic notions and facts from the theory of cooperative interval games. We study the properties of the interval Shapley value on the class of size monotonic interval games in Section 3. Finally, Section 4 gives an axiomatic characterization of the interval Shapley value on a special subclass of cooperative interval games.

2 Preliminaries

In this section some preliminaries from interval calculus and the theory of cooperative interval games are given (Alparslan Gök, Branzei and Tijs (2008) and Alparslan Gök, Miquel and Tijs (2009)).

A *cooperative interval game* is an ordered pair $< N, w >$ where $N = \{1, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all nonempty, compact intervals in $\mathbb{R}$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition S in the interval game $< N, w >$ is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the minimal reward which coalition S could receive on its own and $\overline{w}(S)$ is the maximal reward which coalition S could get. The family of all interval games with player set N is denoted by IG^N. We denote by $I(\mathbb{R})^N$ the set of all such interval payoff vectors.

Let $I, J \in I(\mathbb{R})$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then,

(i) $I + J = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$;

(ii) $\alpha I = \left[\alpha\underline{I}, \alpha\overline{I}\right]$.

By (i) and (ii) we see that $I(\mathbb{R})$ has a cone structure.

In this paper we also need a partial substraction operator. We define $I - J$, only if $|I| \geq |J|$, by $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$. Note that $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. We recall that I is weakly better than J, which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. We also use the reverse notation $I \preccurlyeq J$, if and only if $\underline{I} \leq \underline{J}$ and $\overline{I} \leq \overline{J}$.

For $w_1, w_2 \in IG^N$ we say that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$, for each $S \in 2^N$. For $w_1, w_2 \in IG^N$ and $\lambda \in \mathbb{R}_+$ we define $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$ for each $S \in 2^N$. So, we conclude that IG^N endowed with $\preccurlyeq$ is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by $(w_1 - w_2)(S) = w_1(S) - w_2(S)$.

The model of interval cooperative games is an extension of the model of classical TU-games. We recall that a classical TU-game $< N, v >$ is defined by $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. We denote the family of such games by G^N, and recall that G^N is a $(2^{|N|} - 1)$- dimensional linear space for which unanimity games form an interesting basis. Let $S \in 2^N \setminus \{\emptyset\}$. The unanimity game based on S, $u_S : 2^N \to \mathbb{R}$ is defined by

$$u_S(T) = \begin{cases} 1, & S \subset T \\ 0, & \text{otherwise.} \end{cases}$$

The reader is referred to Part I in Branzei, Dimitrov and Tijs (2008) for a survey on classical TU-games.

Now, we recall that the interval imputation set $\mathcal{I}(w)$ of the interval game w, is defined by

$$\mathcal{I}(w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = w(N), w(i) \preccurlyeq I_i, \text{ for all } i \in N \right\},$$

and the interval core $\mathcal{C}(w)$ of the interval game w, is defined by

$$\mathcal{C}(w) = \left\{ (I_1, \ldots, I_n) \in \mathcal{I}(w) | \sum_{i \in S} I_i \succcurlyeq w(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

We notice that both $\mathcal{I}(w)$ and $\mathcal{C}(w)$ consist of efficient interval payoff vectors, i.e. $(I_1, \ldots, I_n) \in I(\mathbb{R})^N$ with $\sum_{i \in N} I_i = w(N)$, satisfying specific rationality conditions.

We call a game $< N, w >$ *size monotonic* if $< N, |w| >$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMIG^N$ the class of size monotonic interval games with player set N.
The interval marginal operators and the interval Shapley value were defined on $SMIG^N$ in Alparslan Gök, Branzei and Tijs (2008) as follows.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \to N$ of N. The *interval marginal operator* $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ corresponding to σ, associates with each $w \in SMIG^N$ the *interval marginal vector* $m^\sigma(w)$ of w with respect to σ defined by $m_i^\sigma(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i))$ for each $i \in N$, where $P_\sigma(i) := \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$, and $\sigma^{-1}(i)$ denotes the entrance number of player i. We notice that $m^\sigma(w)$ is an efficient interval payoff vector for each $\sigma \in \Pi(N)$. For size monotonic games $< N, w >$, $w(T) - w(S)$ is defined for all $S, T \in 2^N$ with $S \subset T$ since $|w(T)| = |w|(T) \geq |w|(S) = |w(S)|$. Now, we notice that for each $w \in SMIG^N$ the interval marginal vectors $m^\sigma(w)$ are defined for each $\sigma \in \Pi(N)$, because the monotonicity of $|w|$ implies $\overline{w}(S \cup \{i\}) - \underline{w}(S \cup \{i\}) \geq \overline{w}(S) - \underline{w}(S)$, which can be rewritten as $\overline{w}(S \cup \{i\}) - \overline{w}(S) \geq \underline{w}(S \cup \{i\}) - \underline{w}(S)$. So, $w(S \cup \{i\}) - w(S)$ is defined for each $S \subset N$ and $i \notin S$.

The *interval Shapley value* $\Phi : SMIG^N \to I(\mathbb{R})^N$ is defined by

$$\Phi(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N. \quad (1)$$

We can write (1) as follows

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))). \quad (2)$$

The terms after the summation sign in (2) are of the form $w(S \cup \{i\}) - w(S)$, where S is a subset of N not containing i.

Note that there are exactly $|S|!(n-1-|S|)!$ orderings for which one has $P^\sigma(\{i\}) = S$. The first factor, $|S|!$, corresponds to the number of orderings of S and the second factor, $(n-1-|S|)!$, is just the number of orderings of $N \setminus (S \cup \{i\})$. Using this, we can rewrite (2) as

$$\Phi_i(w) = \sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!} (w(S \cup \{i\}) - w(S))). \quad (3)$$

Note that

$$\sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!} = 1. \quad (4)$$

3 Properties of the interval Shapley value

In this section we study some properties of the interval Shapley value on the class of size monotonic interval games.

PROPOSITION 3.1. *The interval Shapley value* $\Phi : SMIG^N \to I(\mathbb{R})^N$ *is additive.*

Proof. First, we show that for each $\sigma \in \Pi(N)$ the interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ is additive, i.e., for all $w_1, w_2 \in SMIG^N$, $m^\sigma(w_1 + w_2) = m^\sigma(w_1) + m^\sigma(w_2)$.
Let $\sigma \in \Pi(N)$ and $k \in N$. Then,

$$\begin{aligned} m^\sigma_{\sigma(k)}(w_1 + w_2) &= (w_1 + w_2)(\sigma(1), \dots, \sigma(k)) \\ &- (w_1 + w_2)(\sigma(1), \dots, \sigma(k-1)) \\ &= w_1(\sigma(1), \dots, \sigma(k)) - w_1(\sigma(1), \dots, \sigma(k-1)) \\ &+ w_2(\sigma(1), \dots, \sigma(k)) - w_2(\sigma(1), \dots, \sigma(k-1)) \\ &= m^\sigma_{\sigma(k)}(w_1) + m^\sigma_{\sigma(k)}(w_2). \end{aligned}$$

Now, using the additivity property of interval marginal operators we obtain that $\Phi : SMIG^N \to I(\mathbb{R})^N$ is an *additive* map, i.e.,

$$\begin{aligned} \Phi(w_1 + w_2) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w_1 + w_2) \\ &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w_1) + \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w_2) \\ &= \Phi(w_1) + \Phi(w_2), \end{aligned}$$

for all $w_1, w_2 \in SMIG^N$. □

Let $w \in SMIG^N$ and $i, j \in N$. Then, i and j are called *symmetric players*, if $w(S \cup \{j\}) - w(S) = w(S \cup \{i\}) - w(S)$, for each S with $i, j \notin S$. We leave the proof of the following proposition to the reader.

PROPOSITION 3.2. *Let* $i, j \in N$ *be symmetric players in* $w \in SMIG^N$*. Then,* $\Phi_i(w) = \Phi_j(w)$.

Let $w \in SMIG^N$ and $i \in N$. Then, i is called a *dummy player* if $w(S \cup \{i\}) = w(S) + w(\{i\})$, for each $S \in 2^{N \setminus \{i\}}$.

Proposition 3.3. *The interval Shapley value* $\Phi : SMIG^N \to I(\mathbb{R})^N$ *has the dummy player property, i.e.* $\Phi_i(w) = w(\{i\})$ *for all* $w \in SMIG^N$ *and for all dummy players* i *in* w.

Proof. This follows from (3) by taking (4) into account. □

Proposition 3.4. *The interval Shapley value* $\Phi : SMIG^N \to I(\mathbb{R})^N$ *is efficient, i.e.,* $\sum_{i\in N} \Phi_i(w) = w(N)$.

Proof. First, we show that for each $\sigma \in \Pi(N)$ the interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ is efficient, i.e. $\sum_{i\in N} m_i^\sigma(w) = w(N)$.
Let $w \in SMIG^N$ and $\sigma \in \Pi(N)$. Then,

$$\begin{aligned} \sum_{i\in N} m_i^\sigma(w) &= \sum_{k=1}^{N} m_{\sigma(k)}^\sigma(w) \\ &= w(\sigma(1)) + \sum_{k=2}^{n} w(\sigma(1),\ldots,\sigma(k)) - w(\sigma(1),\ldots,\sigma(k-1)) \\ &= w(\sigma(1)) + w(\sigma(1),\ldots,\sigma(n)) - w(\sigma(1)) = w(N). \end{aligned}$$

Now, using the efficiency of interval marginal operators, we obtain that $\Phi : SMIG^N \to I(\mathbb{R})^N$ is an efficient map, i.e.,

$$\sum_{i\in N} \Phi_i(w) = \frac{1}{n!}\sum_{i\in N}\sum_{\sigma\in\Pi(N)} m_i^\sigma(w) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)}\sum_{i\in N} m_i^\sigma(w) = \frac{1}{n!} n! w(N) = w(N),$$

for each $w \in SMIG^N$. □

4 An axiomatization of the interval Shapley value

Let $S \in 2^N \setminus \{\emptyset\}$, $I \in I(\mathbb{R})$ and let u_S be the unanimity game based on S. The cooperative interval game $< N, Iu_S >$ is defined by $(Iu_S)(T) = u_S(T)I$ for each $T \in 2^N \setminus \{\emptyset\}$, and its Shapley value is given by

$$\Phi_i(Iu_S) = \begin{cases} I/|S|, & i \in S \\ [0,0], & i \notin S. \end{cases}$$

In the sequel such interval games will play a central role. We denote by KIG^N the additive cone generated by the set

$$K = \left\{ I_S u_S | S \in 2^N \setminus \{\emptyset\} \right\},$$

where $I_S \in I(\mathbb{R})$. So, each element of the cone is a finite sum of elements of K. We notice that $KIG^N \subset SMIG^N$, and axiomatically characterize the restriction of the interval Shapley value to the cone KIG^N .

Theorem 4.1. *There is a unique solution $\Psi : KIG^N \to I(\mathbb{R})^N$ satisfying the properties of additivity, efficiency, dummy-player and symmetry. This solution is the interval Shapley value.*

Proof. From Propositions 3.1, 3.2, 3.3 and 3.4 and $KIG^N \subset SMIG^N$ we obtain that Φ satisfies the four properties on KIG^N.
Conversely, let Ψ be an interval value satisfiying the four properties on KIG^N. We have to show that $\Psi = \Phi$. Take $w \in KIG^N$. Notice that w can be written as $w = \sum_{S \in 2^N \setminus \{\emptyset\}} I_S u_S$. Then, for each $S \in 2^N \setminus \{\emptyset\}$ and $I_S \in I(\mathbb{R})$ we have $\Psi(\sum_{S \in 2^N \setminus \{\emptyset\}} I_S u_S) = \sum_{S \in 2^N \setminus \{\emptyset\}} \Psi(I_S u_S)$ and $\Phi(\sum_{S \in 2^N \setminus \{\emptyset\}} I_S u_S) = \sum_{S \in 2^N \setminus \{\emptyset\}} \Phi(I_S u_S)$ by additivity. Therefore we need to show that for each $S \in 2^N \setminus \{\emptyset\}$ and $I_S \in I(\mathbb{R})$, $\Psi(I_S u_S) = \Phi(I_S u_S)$. Take $S \in 2^N \setminus \{\emptyset\}$ and $I_S \in I(\mathbb{R})$. Note that for all $i \in N \setminus S$,

$$(I_S u_S)(T \cup \{i\}) - (I_S u_S)(T) = [0, 0] = (I_S u_S)(\{i\}).$$

By the dummy player property, we have

$$\Psi_i(I_S u_S) = \Phi_i(I_S u_S) = [0, 0], \text{for all } i \in N \setminus S. \quad (5)$$

Now, suppose that $i, j \in S$, $i \neq j$. Then the symmetry property implies that

$$\Phi_i(I_S u_S) = \Phi_j(I_S u_S) \text{ for all } i, j \in S, \quad (6)$$

and, similarly,

$$\Psi_i(I_S u_S) = \Psi_j(I_S u_S) \text{ for all } i, j \in S.$$

By efficieny, (5) and (6) we obtain for any $S \in 2^N \setminus \{\emptyset\}$ and $I_S \in I(\mathbb{R})$

$$\Psi_i(I_S u_S) = \Phi_i(I_S u_S) = \frac{I_S}{|S|}. \quad (7)$$

Hence, $\Psi(w) = \Phi(w)$ for all $w \in KIG^N$ by (5) and (7). □

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